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**Universality and non-universality
in the Ashkin–Teller model**

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1. Introduction.

Two dimensional classical spin systems play a very special role in statistical mechanics, in providing the simplest non trivial examples of systems undergoing a phase transition.

The first of these model to be extensively studied was the Ising model, [Pe][O][Ka49][KO][Ya] whose importance relies in the fact that it first gave firm and quantitative indications that a microscopic short range interaction can produce phase transitions which deeply differ from that described by mean field approximation.

In the Ising model many detailed informations about the microscopic structure of the phases in the low or high temperature regime can be obtained by perturbative techniques (cluster expansion [Ru63][GM68][D][Ru69]), by correlation inequalities [Gr][FKG][Le74] and by probabilistic methods (*e.g.* the “infinite cluster” method [Ru79][Ai80][Hi]) and some of the critical properties can be deduced by combination of the previous techniques together with the use of “infrared bounds” [Fr][Ai82]. However, most of the results about the behaviour of thermodynamic functions near the critical temperature rely on the exact solution, first obtained by Onsager and after him reproduced in many different independent ways [KO][KWa][SML][H][S].

The Ising model in zero magnetic field is solvable in a very strong sense: it can be exactly mapped into a system of free fermions [SML][H][S] and, as a consequence, not only one can calculate the free energy and the magnetization, but exact formulae for many important spin correlation functions can be derived, and the asymptotic behaviour for large distances of some of them can be exactly computed¹. For instance the energy–energy correlation functions can be computed, as well as the asymptotic behaviour of the spin–spin correlation function [MPW][BMW][TM][WMTB][MW] and of some multi–spin correlation functions, when we let the relative distances of the positions of the spins diverge in some special way and directions (*e.g.* along the same horizontal line [Ka69]).

These results allow to calculate the critical exponents, as defined in the usual scaling theory of critical phenomena, and to verify that, even if the scaling laws are all satisfied, as expected, the 2D Ising model exponents are *different* from those expected from Curie–Weiss theory: one says that the Ising model belongs to a different *universality class*.

The development of Renormalization Group [Ka66][DJ][C][Sy][W1][W2] [WF], starting from the end of the 60’s, clarified the concept of universality class, and gave a fundamental explanation to the phenomenological expectation that different models, even describing completely different physical situations, could show the same critical behaviour, in the sense that their critical exponents are the same (if one suitably identifies the corresponding thermodynamic functions in the two systems). In the context of statistical mechanics, it became clear that two systems, with the same symmetries and with interactions differing only by *irrelevant* terms have correlation functions that, at the critical point, show the same asymptotic behaviour in the limit of large distances; that is the two systems have the same critical exponents.

Independently from Renormalization Group, and approximatively at the same time, a new important branch of statistical mechanics arose, that of exactly solvable models, for a review see [Ba82]. In this context, and more specifically in that of 2D spin systems, many explicit examples were constructed of new and unexpected universality classes, different from Ising’s. We refer in particular to two dimensional 6 vertex

¹ It must be stressed that these informations cannot be trivially derived from the exact expression of the free energy, and hard work together with amazing algebraic cancellations are needed for the computation of the asymptotics of correlation functions, even for the “simple” spin–spin correlation function along the same horizontal line, see [MW].

(6V) and 8 vertex (8V) models². The class of 6V models includes the ice model, first solved by Lieb [L1], the F-model and the KDP-model, solved in rapid succession after the ice-model exact solution [L2][L3][Su]; see [LW] for a review on the 6V models. The Lieb’s solution was a breakthrough in statistical mechanics, both because first showed the existence of exactly solvable models other than Ising itself, and because concretely showed the existence of many new universality classes different from Ising’s in the context of 2D spin systems. The latter point was of great importance for the development of the theory of critical phenomena: in fact at the time of the solution of the ice model the universality theory of critical point singularity was not yet developed in its final form. So, when Renormalization Group approach arose around 1969, the 6V models appeared as a counterexample to the universality that Renormalization Group was supposed to predict: depending on the specific choices of the energies assigned to the different vertex configurations one could find different values for the critical exponents.

This fact, not well understood at the beginning by a fundamental point of view, was dismissed by the theoretical physics community on the grounds that the 6V models are spin model “with constraints” (see footnote 2), that is too pathological to be well described by the universality theory of critical phenomena.

However the deep meaning of Lieb’s counterexamples was made clear by Baxter’s exact solution of the 8V model, contained in a series of papers from 1971 to 1977 [Ba]³: it made clear to everybody that the 6V models could not be considered as pathological counterexamples. As remarked in footnote 2, 8V models are genuine short range Ising models with finite interaction and one can for instance consider a path in the parameters space continuously linking two 6V models defined by different choices of the energies associated with the vertex configurations. The remarkable result following by the 8V solution is that along this path the 8V critical exponents change *continuously*, and continuously connect those of the two different 6V models.

This observation was crucial and led to a much better understanding of the theories that were put forward to explain critical phenomena, first among all Renormalization Group itself. In modern language the solution of the above “paradoxes” relies on the fact that the 6V and 8V models with different choices of parameters differ by *marginal* terms: however this fact is not apparent in the original spin variable, and in order to realize this one has to reformulate all this models as suitable field theory models (that is not an easy task).

Even if many important informations about the thermodynamics of vertex models can be found from their exact solution, these models are exactly solvable in a sense much weaker than that of Ising.

The 6V models are solvable by Bethe ansatz, that is by assuming that the eigenvector of the transfer matrix with largest eigenvalue is a linear combination of plane waves; and calculating the coefficients of the linear combination by solving a (complicated) integral equation. This allows to find an exact expression for the free energy $f(\beta, E)$, as a function of the temperature β^{-1} and of an external electric field E (so that by computing the derivatives of f w.r.t. E one can study the critical behaviour of the electric response function); but nothing can be said about more complicated correlation functions, it is not even possible to write formal expression for them.

The solution of the 8V model is even more involved and sophisticated and is based on a reformulation of the problem of calculating the free energy into the problem of solving a set of coupled elliptic integral equations (the so called Yang–Baxter triangle–star equations). Also in this case it is not possible to find

² The vertex models are defined by associating a direction to each of the bonds linking the sites of a 2D lattice; and by allowing only a few configurations of the arrows entering or exiting a lattice site. In the 6V (8V) models only 6 (8) different configurations are allowed at each site, and different energies are assigned to each allowed configuration. The 8V model can be easily mapped into 2D spin models, described by two Ising layers, coupled by a 4 spin interaction. The 6V models can be obtained from the spin description of 8V by letting the coupling constants tending to infinity in some specific way (they can be considered as Ising models “with constraints”).

³ Baxter’s solution represents one of the major achievements of mathematical physics in the 1970’s: it introduced for the first time in theoretical physics the use of triangle–star equations and of corner transfer matrix, which are nowadays fundamental tools for the study of quantum groups and integrable systems.

(even formal) expressions for generic correlation functions, but only the free energy as a function of some thermodynamic parameters can be calculated, so that only informations about special low order correlation functions can be obtained.

Relying the solutions of 6V and 8V on the explicit analytic solution of special integral equations, it is not surprising that even small and apparently harmless modifications (from the Renormalization Group point of view) of these models completely destroy their integrability. Also, the exact solutions do not give any information about the thermodynamic behaviour of systems obtained as small perturbations of 6V and 8V.

On the other hand one can hope that many relevant properties of the integrable models are quite robust under perturbations. Indeed, on the basis of operator algebra and scaling theory, it was conjectured since a long time that a universality property holds for Ising, in the sense that by adding to it, for instance, a next to nearest neighbor interaction, the critical indexes remain unchanged. A similar universality property was conjectured for the 8V model. By scaling theory arguments, Kadanoff [Ka77] found evidence that 8V is in the same class of universality of the Ashkin–Teller model ⁴, in the sense that the critical exponents are the same, if one suitably identifies the coupling constants. Further evidence for this conclusion was given in [PB], by second order Renormalization Group, and in [LP][N], by a heuristic mapping of both the 8V and the Ashkin–Teller models into the massive Luttinger model, a not integrable model describing massive interacting fermions on the continuum in 1+1 dimensions.

As suggested by the previous discussion, the natural method to relate non-integrable models to integrable ones is given by Renormalization Group (RG). This was realized long ago, but the main open problem in this context was to implement RG in a rigorous way; and, even at a heuristic level, to understand in a detailed and quantitative way from the RG point of view how the crossovers between the different universality classes are realized, when one let continuously vary the strength of the coupling constants defining the interaction among spins.

In this dissertation we want to describe a constructive method for studying thermodynamic and correlation functions at the critical point for a wide class of two dimensional classical spin systems, obtained as perturbations of the Ising model, including the next to nearest neighbor Ising, the 8V model and Ashkin–Teller. The method was first introduced in [PS] and [M] and is based on an exact mapping of the spin model into a model of interacting spinless fermions in 1+1 dimensions and on the implementation of constructive fermionic Renormalization Group methods for the construction of the effective potential and of the correlation functions. The constructive fermionic Renormalization Group methods we apply were developed by the Roma’s school in the last decade [BG1][BGPS][BoM][GS][BM] and are technically based on the so-called functional renormalization group, developed in the 1980’s starting from [Po] [GN], see [G1][BG] for reviews.

We will apply the method to the analysis of the critical behaviour of the specific heat C_v in the *Ashkin–Teller* model and we will rigorously prove an old conjecture by Baxter and Kadanoff about the critical behaviour of Ashkin–Teller (AT), in correspondence of different choices of the parameters defining the model (the inter-layer interaction λ and the anisotropy $J^{(1)} - J^{(2)}$, see (1.1)). We shall study in detail how the crossover between the different universality classes is realized when we let $J^{(1)} - J^{(2)} \rightarrow 0$ and how the location of the critical points is renormalized by the interaction λ , in the region of small λ .

1.1. The Ashkin–Teller model.

The Ashkin–Teller model [AT] was introduced as a generalization of the Ising model to a four component system; in each site of a bidimensional lattice there is a spin which can take four values, and only nearest neighbor spins interact. The model can be also considered a generalization of the four state Potts model to which it reduces for a suitable choice of the parameters.

⁴ Ashkin–Teller (AT) is defined as a pair of Ising layers coupled via a four spin plaquette interaction, different from that of 8V; AT is not integrable and, in correspondence of some special choices of its parameters, it reduces to Ising and to the 4-states Potts model.

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A very convenient representation of the Ashkin–Teller model is in terms of Ising spins [F]: given a square sublattice $\Lambda_M \subset \mathbb{Z}^2$ of side M , one associates at each site $\mathbf{x} \in \Lambda_M$ two kinds of Ising spins, $\sigma_{\mathbf{x}}^{(1)}$, $\sigma_{\mathbf{x}}^{(2)}$, assuming two possible values ± 1 . The AT Hamiltonian is assumed to be:

$$H_{\Lambda_M}^{AT} = - \sum_{\langle \mathbf{x}, \mathbf{y} \rangle \in \Lambda_M} \left[J^{(1)} \sigma_{\mathbf{x}}^{(1)} \sigma_{\mathbf{y}}^{(1)} + J^{(2)} \sigma_{\mathbf{x}}^{(2)} \sigma_{\mathbf{y}}^{(2)} + \lambda \sigma_{\mathbf{x}}^{(1)} \sigma_{\mathbf{y}}^{(1)} \sigma_{\mathbf{x}}^{(2)} \sigma_{\mathbf{y}}^{(2)} \right] \equiv \sum_{\mathbf{x} \in \Lambda_M} H_{\mathbf{x}}^{AT}, \quad (1.1)$$

where \mathbf{x}, \mathbf{y} are nearest neighbor sites and the last identity is a definition for $H_{\mathbf{x}}^{AT}$. Periodic boundary conditions will be assumed throughout the work.

The case in which the two Ising subsystems are identical $J^{(1)} = J^{(2)}$ is called *isotropic*, the opposite case *anisotropic*.

When the coupling λ is $= 0$, Ashkin–Teller (AT) reduces to two independent Ising models and it has of course *two* critical temperatures if $J^{(1)} \neq J^{(2)}$.

When $J^{(1)} = J^{(2)} = \lambda$, AT reduces to the four states Potts model.

We shall study the case $J^{(i)} > 0$, $i = 1, 2$, that is the case in which the two Ising subsystems are *ferromagnetic*.

AT is a model for a number of 2d magnetic compounds: for instance layers of atoms and molecules adsorbed on clean surfaces, like selenium on nichel, molecular oxygen on graphite, atomic oxygen on tungsten; and layers of oxygen atoms in the basal Cu–O plane of some cuprates, like $\text{YBa}_2\text{Cu}_3\text{O}_z$, are believed to constitute physical realizations of the AT model [DR][Bak][Bar]. Theoretical results on AT can give detailed informations on the critical behaviour and the phase diagrams of such systems, which can be experimentally measured by means of electron diffraction techniques.

Also, as explained in previous section, the importance of AT is in providing a conceptual laboratory in which the highly non trivial phenomenon of phase transitions can be understood quantitatively in a relatively manageable model; in particular it has attracted great theoretical interest because is a simple and non trivial generalization of the Ising and four-state Potts models, showing a rich variety of critical behaviours, depending on the choices of the parameters $J^{(i)}$ and λ in (1.1). AT is not exactly solvable, except in the trivial $\lambda = 0$ case, and it has great theoretical interest to develop techniques that, *without any use of exact solutions*, could allow to understand the AT critical behaviour. In fact exact solutions are quite rare and generally peculiar of low dimensions, while RG methods are expected to work in much more general situations: then it is important to refine RG techniques in a simple but non trivial playground, as that offered by AT.

The thermodynamic behaviour of the anisotropic AT model is not well understood even at a heuristic level. What is “known” is mainly based on conjectures, suggested by scaling theory, and on numerics.

A first conjecture, proposed by Wu and Lin [WL], concerns the critical points: from the symmetries of the model, it is expected that AT, even in the interacting case (*i.e.* $\lambda \neq 0$), has *two* critical temperatures for $J^{(1)} \neq J^{(2)}$ which coincide at the isotropic point $J^{(1)} = J^{(2)}$. However nothing has been proposed about the location of the critical points, even at a conjectural level.

Kadanoff [Ka77] and Baxter [Ba82] conjectured that the critical properties in the anisotropic and in the isotropic case are completely different; in the first case the critical behaviour should be described in terms of *universal* critical indices (identical to those of the 2D Ising model) while in the isotropic case the critical behaviour should be *nonuniversal* and described in terms of indexes which are non trivial functions of λ . In other words, the AT model should exhibit a *universal–nonuniversal* crossover when the isotropic point is reached.

The general anisotropic case was studied numerically by Migdal–Kadanoff Renormalization Group [DR], Mean Field Approximation and Monte Carlo [Be], real-space Renormalization Group [Bez] Transfer Matrix Finite–Size–Scaling [Bad]; such results give evidence of the fact that, far away from the isotropic point, AT has two critical points and belongs to the same universality class of the Ising model but give essentially no informations on the critical behaviour when the anisotropy is small. The problem of how the crossover from

universal to nonuniversal behaviour is realized in the isotropic limit remained for years completely unsolved, even at a heuristic level.

1.2. Results.

Our main results concern the analytical properties of the free energy in an interval of temperatures around the critical temperatures; and the critical behaviour of the specific heat. These thermodynamic quantities are defined in the usual way: if β is the inverse temperature, the partition function at finite volume is:

$$\Xi_{\Lambda_M} \stackrel{def}{=} \sum_{\sigma_{\Lambda_M}^{(1)}, \sigma_{\Lambda_M}^{(2)}} e^{-\beta H_{\Lambda_M}^{AT}}, \quad (1.2)$$

where $\sigma_{\Lambda_M}^{(i)} \equiv \{\sigma_{\mathbf{x}}^{(i)} \mid \mathbf{x} \in \Lambda_M\}$; correspondingly, the free energy and the specific heat are defined as:

$$f = -\frac{1}{\beta} \lim_{M \rightarrow \infty} \frac{1}{M^2} \log \Xi_{\Lambda_M} \quad , \quad C_v = \lim_{M \rightarrow \infty} \frac{\beta^2}{M^2} \sum_{\mathbf{x}, \mathbf{y} \in \Lambda_M} \langle H_{\mathbf{x}}^{AT} H_{\mathbf{y}}^{AT} \rangle_{\Lambda_M, T}, \quad (1.3)$$

where $\langle \cdot \rangle_{\Lambda_M, T}$ denotes the truncated expectation w.r.t. the Gibbs distribution with Hamiltonian (1.1).

We find convenient to introduce the variables

$$t = \frac{t^{(1)} + t^{(2)}}{2}, \quad u = \frac{t^{(1)} - t^{(2)}}{2} \quad (1.4)$$

with $t^{(j)} = \tanh \beta J^{(j)}$, $j = 1, 2$. The parameter t has the role of a *reduced temperature* and u measures the *anisotropy* of the system. We shall consider the free energy or the specific heat as functions of t, u, λ . When $\lambda = 0$ the model (1.1) reduces to a pair of decoupled Ising models and the specific heat C_v can be immediately computed from the Ising model exact solution; the system admits two critical points, defined by

$$\tanh \beta J^{(i)} = \sqrt{2} - 1, \quad i = 1, 2, \quad (1.5)$$

or, in terms of the parameters t, u defined in (1.4):

$$t_c^\pm = \sqrt{2} - 1 \pm |u|. \quad (1.6)$$

As it is well-known from Ising's exact solution, near the two critical temperatures the specific heat shows a logarithmic divergence: $C_v \simeq -C \log |t - t_c^\pm|$, where $C > 0$.

Consider now the $\lambda \neq 0$ case. If the anisotropy is strong the two Ising subsystems have very different critical temperatures: so, if the temperature of the coupled system is near to the critical temperature of one of the Ising subsystems, one can expect that AT is essentially equivalent to a single critical Ising model, perturbed by a small "random noise", produced by the non-critical fluctuations of the second Ising subsystem; in such a case one expects that the effect of the coupling is at most that of changing the value of the critical temperatures [PS]⁵. On the other hand if the anisotropy is small the two system will become critical almost at the same temperature and the properties of the system could change drastically.

⁵ Note that, because of the structure of the Hamiltonian (1.1) (in which the interaction has the form of a product of bond interactions), this heuristic picture applies both to the case the non critical Ising model is well inside the paramagnetic phase and to the case it is well inside the magnetized phase: in both cases, if the system 2 is the system far from criticality, we can rewrite $\sigma_{\mathbf{x}}^{(2)}$ as $\sigma_{\mathbf{x}}^{(2)} = m_2^* + \delta\sigma_{\mathbf{x}}^{(2)}$, where m_2^* is the (unperturbed) magnetization of system 2, and $\delta\sigma_{\mathbf{x}}^{(2)}$ is the field associated with the non critical fluctuations of $\sigma_{\mathbf{x}}^{(2)}$ around its average value; one can then expect that the effect of the interaction of system 1 with system 2 is just that of changing the coupling $J^{(1)}$ into an effective coupling $J^{(1)} + \lambda(m_2^*)^2 + \delta J$, where δJ is a small random noise, generated by the non-critical fluctuations of $\sigma^{(2)}$ around its average value. Since we shall assume $J^{(1)}$ to be $O(1)$, it makes no qualitative difference whether m_2^* is vanishing or not.

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With the notations introduced above and calling D a sufficiently small $O(1)$ interval (*i.e.* with amplitude independent of λ) centered around $\sqrt{2} - 1$, we can express our main result as follows [GM1][GM2].

THEOREM. *There exists $\varepsilon > 0$ such that, if $|\lambda| \leq \varepsilon$ and $t \pm u \in D$, the AT model admits two critical points of the form:*

$$t_c^\pm(\lambda, u) = \sqrt{2} - 1 + \nu(\lambda) \pm |u|^{1+\eta}(1 + \delta(\lambda, u)) . \quad (1.7)$$

Here ν and δ are $O(\lambda)$ corrections and $\eta = \eta(\lambda) = -b\lambda + O(\lambda^2)$, $b > 0$, is an analytic function of λ . If $|\lambda| \leq \varepsilon$, $t \pm u \in D$ and $t \neq t_c^\pm$, the free energy and the specific heat of the model are analytic in λ, t, u ; in the same region of parameters, the specific heat C_v can be written as:

$$C_v = F_1 \Delta^{2\eta_c} \log \frac{|t - t_c^-| \cdot |t - t_c^+|}{\Delta^2} + F_2 \frac{1 - \Delta^{2\eta_c}}{\eta_c} + F_3 , \quad (1.8)$$

where: $2\Delta^2 = (t - t_c^-)^2 + (t - t_c^+)^2$; $\eta_c = a\lambda + O(\lambda^2)$, $a \neq 0$; and F_1, F_2, F_3 are functions of t, u, λ , bounded above and below by $O(1)$ constants.

A first interesting result that can be read from the Theorem is that the location of the critical points is dramatically changed by the interaction, see (1.7). The difference of the interacting critical temperatures normalized with the free one $G(\lambda, u) \equiv (t_c^+(\lambda, u) - t_c^-(\lambda, u)) / (t_c^+(0, u) - t_c^-(0, u))$ rescales with the anisotropy parameter as a power law $\sim |u|^\eta$, and in the limit $u \rightarrow 0$ it vanishes or diverges, depending on the sign of λ (this is because $\eta = -b\lambda + O(\lambda^2)$, with $b > 0$). In Fig. 1 we plot the qualitative behaviour of $G(\lambda, u)$ as a function of u , for two different values of λ (*i.e.* we plot the function u^η , with $\eta = 0.3, -0.3$ respectively).

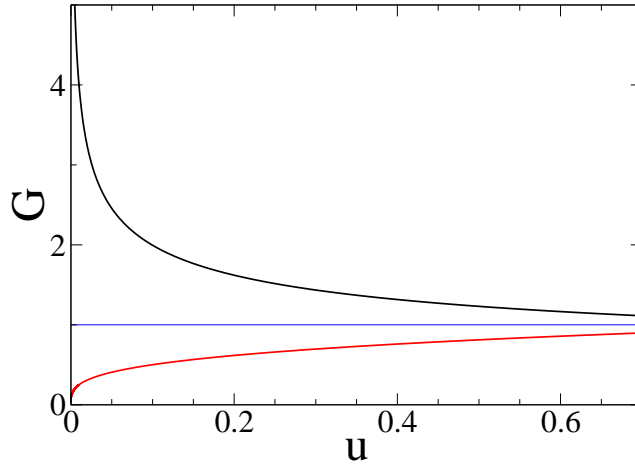


FIG. 2. The behaviour of the difference G between the interacting critical temperatures normalized to the free one, for two different values of λ ; depending on the sign of the interaction, it diverges or vanishes in the isotropic limit.

As far as we know, the existence of the critical index $\eta(\lambda)$ was not known in the literature, even at a heuristic level.

From (1.8) it follows that there is universality for the specific heat, in the sense that it diverges logarithmically at the critical points, as in the Ising model. However the coefficient of the log is *anomalous*: in fact if t is near to one of the critical temperatures $\Delta \simeq \sqrt{2}|u|^{1+\eta}$ so that the coefficient in front of the logarithm behaves like $\sim |u|^{2(1+\eta)\eta_c}$, with η_c a new anomalous exponent $O(\lambda)$; in particular it is vanishing or diverging as $u \rightarrow 0$ depending on the sign of λ . We can say that the system shows an *anomalous universality* which is a sort of new paradigmatic behaviour: the singularity at the critical points is described in terms of universal critical indexes and nevertheless, in the isotropic limit $u \rightarrow 0$, some quantities, like the difference of the critical temperatures and the constant in front of the logarithm in the specific heat, scale with anomalous critical indexes, and they vanish or diverge, depending on the sign of λ .

Eq(1.8) clarifies how the universality–nonuniversality crossover is realized as $u \rightarrow 0$. When $u \neq 0$ only the first term in eq(1.8) can be log–singular in correspondence of the two critical points; however the logarithmic term dominates on the second one only if t varies inside an extremely small region $O(|u|^{1+\eta}e^{-c/|\lambda|})$ around the critical points (here c is a positive $O(1)$ constant). Outside such region the power law behaviour corresponding to the second addend dominates. When $u \rightarrow 0$ one recovers the power law decay first found by Mastropietro [M] in the isotropic case:

$$C_v \simeq F_2 \frac{1 - |t - t_c|^{2\eta_c}}{\eta_c} \quad (1.9)$$

In Fig. 2 we plot the qualitative behaviour of C_v as a function of t . The three graphs are plots of eq(1.8), with $F_1 = F_2 = 1$, $F_3 = 0$, $u = 0.01$, $\eta = \eta_c = 0.1, 0, -0.1$ respectively; the central curve corresponds to the case $\eta = 0$, the upper one to $\eta < 0$ and the lower to $\eta > 0$.

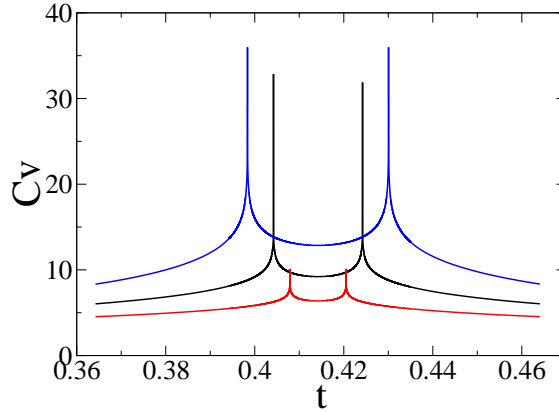


FIG. 2. The behaviour of the specific heat C_v for three different values of λ , showing the log-singularities at the critical points; in the isotropic limit the two critical points tend to coincide, the lower curve becomes continuous while the upper develops a power law divergence.

It now worths to make some technical remarks about the Theorem above.

The first is about the range of parameters where the Theorem holds. The key hypothesis for the validity of the Theorem is the smallness of λ . When $\lambda = 0$ the critical points correspond to $t \pm u = \sqrt{2} - 1$: hence for simplicity we restrict $t \pm u$ in a sufficiently small $O(1)$ interval around $\sqrt{2} - 1$. A possible explicit choice for D , convenient for our proof, could be $D = [\frac{3(\sqrt{2}-1)}{4}, \frac{5(\sqrt{2}-1)}{4}]$. We expect that our technique would allow us to prove the above theorem, at the cost of a lengthier discussion, for any $t^{(1)}, t^{(2)} > 0$: of course in that

case we should distinguish different regions of parameters and treat in a different way the cases of low or high temperature or the case of big anisotropy (*i.e.* the cases $t \ll \sqrt{2} - 1$ or $t \gg \sqrt{2} - 1$ or $|u| \gg 1$).

The second remark is about the analyticity of the specific heat. It is claimed that C_v is analytic in λ, t, u outside the critical line. However, this is not apparent from (1.8), because Δ is non analytic in u at $u = 0$ (of course the bounded functions F_j are non analytic in u also, in a suitable way compensating the non analyticity of Δ). We get to (1.8) by interpolating two different asymptotic behaviours of C_v in the regions $|t - \bar{t}_c| < 2|u|^{1+\eta}$ and $|t - \bar{t}_c| \geq 2|u|^{1+\eta}$, where \bar{t}_c is the average point between t_c^+ and t_c^- ; then, the non analyticity of Δ is introduced “by hands” by our estimates and it is not intrinsic for C_v . (1.8) is simply a convenient way to describe the crossover between different critical behaviours of C_v .

Finally, it must be stressed that we do not study the free energy directly at $t = t_c^\pm(\lambda, u)$, therefore in order to show that $t = t_c^\pm(\lambda, u)$ is a critical point we must study some thermodynamic property like the specific heat by evaluating it at $t \neq t_c^\pm(\lambda, u)$ and $M = \infty$ and then verify that it has a singular behavior as $t \rightarrow t_c^\pm$. The case t precisely equal to t_c^\pm cannot be discussed at the moment with our techniques, in spite of the uniformity of our bounds as $t \rightarrow t_c^\pm$. The reason is that we write the AT partition function as a sum of 16 different partition functions, differing for boundary terms. Our estimates on each single term are uniform up to the critical point; however, in order to show that the free energy computed with one of the 16 terms is the same as the complete free energy, we need to stay at $t \neq t_c^\pm$: in this case boundary terms are suppressed as $\sim e^{-\kappa M|t - t_c^\pm|}$, $\kappa > 0$, as $M \rightarrow \infty$. If we stay exactly at the critical point cancellations between the 16 terms can be present (as it is well known already from the Ising model exact solution [MW]) and we do not have control on the behaviour of the free energy, as the infinite volume limit is approached. We believe that this is a purely technical difficulty and that it could be solved by a more detailed analysis of the cancellations among the different terms appearing in the Ising’s partition function. Another possibility to study AT directly at the critical point would be to adapt our method to the case of open boundary conditions (where even in the fermionic representation the free energy can be written as the logarithm of a single partition function). The interest of studying the model directly at criticality is linked to the possibility of explicitly studying the finite size corrections to the correlation functions and the approach to their conformal limit.

1.3. Outline of the proof.

The proof of the Theorem above is based on a multiscale analysis of the free energy and of the generating function of the energy–energy correlation functions.

The first step to set up the Renormalization Group machinery is finding a convenient field theory which gives an equivalent description of our spin system. We give a fermionic representation of the theory, following the same strategy of [PS][M]. We start from the well known representation of the Ising model free energy in terms of a sum of *Pfaffians* [MW] which can be equivalently written (see Ref. [ID][S]) as *Grassmann functional integrals*, formally describing massive non interacting Majorana fermions $\psi, \bar{\psi}$ on a lattice with action

$$\sum_{\mathbf{x}} \frac{t}{4} \left[\psi_{\mathbf{x}}(\partial_1 - i\partial_0)\psi_{\mathbf{x}} + \bar{\psi}_{\mathbf{x}}(\partial_1 + i\partial_0)\bar{\psi}_{\mathbf{x}} - 2i\bar{\psi}_{\mathbf{x}}(\partial_1 + \partial_0)\psi_{\mathbf{x}} \right] + i(\sqrt{2} - 1 - t)\bar{\psi}_{\mathbf{x}}\psi_{\mathbf{x}}, \quad (1.10)$$

where ∂_j are discrete derivatives; criticality corresponds to the massless case. If $\lambda = 0$ the free energy and specific heat of the AT model can be written as sum of Grassmann integrals describing *two* kinds of Majorana fields, with masses $m^{(1)} = t^{(1)} - \sqrt{2} + 1$ and $m^{(2)} = t^{(2)} - \sqrt{2} + 1$.

If $\lambda \neq 0$ again the free energy and the specific heat can be written as Grassmann integrals, but the Majorana fields are *interacting* with a short range potential. By performing a suitable change of variables [ID][PS][M] and integrating out the ultraviolet degrees of freedom, the effective action can be written as

$$Z_1 \sum_{\mathbf{x}, \omega, \alpha} \left[\psi_{\omega, \mathbf{x}}^+(\partial_1 - i\omega\partial_0)\psi_{\omega, \mathbf{x}}^- - i\omega\sigma_1\psi_{\omega, \mathbf{x}}^+\psi_{-\omega, \mathbf{x}}^- + i\omega\mu_1\psi_{\omega, \mathbf{x}}^\alpha\psi_{-\omega, -\mathbf{x}}^\alpha + \lambda_1\psi_{1, \mathbf{x}}^+\psi_{1, \mathbf{x}}^-\psi_{-1, \mathbf{x}}^+\psi_{-1, \mathbf{x}}^- \right] + \mathcal{W}_1 \quad (1.11)$$

where $\alpha = \pm$ is a *creation-annihilation* index and $\omega = \pm 1$ is a *quasi-particle* index. σ_1 and μ_1 have the role of two *masses* and it holds $\sigma_1 = O(t - \sqrt{2} + 1) + O(\lambda)$, $\mu_1 = O(u)$. \mathcal{W}_1 is a sum of monomials of ψ of arbitrary order, with kernels which are *analytic functions* of λ_1 ; analyticity is a very nontrivial property obtained exploiting anticommutativity properties of Grassman variables via *Gram inequality* for determinants [Le][BGPS]. The ψ^\pm are *Dirac* fields, which are combinations of the Majorana variables $\psi^{(j)}, \bar{\psi}^{(j)}$, $j = 1, 2$, associated with the two Ising subsystems.

One can compute the partition function by expanding the exponential of the action in Taylor series in λ and naively integrating term by term the Grassmann monomials, using the Wick rule; however such a procedure gives poor bounds for the coefficients of this series that, in the thermodynamic limit, can converge only far from the critical points.

In order to study the critical behaviour of the system we perform a multiscale analysis involving non trivial resummations of the perturbative series. The first step is to decompose the propagator $\hat{g}(\mathbf{k})$ as a sum of propagators more and more singular in the infrared region, labeled by an integer $h \leq 1$, so that $\hat{g}(\mathbf{k}) = \sum_{h=-\infty}^1 \hat{g}^{(h)}(\mathbf{k})$, $\hat{g}^{(h)}(\mathbf{k}) \sim \gamma^{-h}$. We compute the Grassmann integrals defining the partition function by iteratively integrating the propagators $\hat{g}^{(1)}, \hat{g}^{(0)}, \dots$. After each integration step we rewrite the partition function in a way similar to the last equation, with $Z_h, \sigma_h, \mu_h, \lambda_h, \mathcal{W}_h$ replacing $Z_1, \sigma_1, \mu_1, \lambda_1, \mathcal{W}_1$, in particular the masses $\sigma_h \pm \mu_h$ and the wave function renormalization Z_h are modified through the iterative scheme; the structure of the action is preserved because of symmetry properties; moreover \mathcal{W}_h is shown to be a sum of monomials of ψ of arbitrary order, with kernels decaying in real space on scale γ^{-h} , which are *analytic functions* of $\{\lambda_h, \dots, \lambda_1\}$, if λ_k are small enough, $k \geq h$, and $|\sigma_k| \gamma^{-k}, |\mu_k| \gamma^{-k} \leq 1$; again analyticity follows from Gram-Hadamard type of bounds.

All the above construction is based on the crucial property that the effective interaction at each scale does not increase: $|\lambda_h| \leq 2|\lambda|$. This property is highly non trivial and at a first naive analysis it even seems false. In fact the effective coupling constants λ_h obey a complicated set of recursive equations, whose right hand side is called, as usual, the *Beta function*. The Beta function can be written as sum of two terms; the first term is common to a wide class of models, including the *Luttinger model*, the *Thirring model*, the *Holstein-Hubbard model* for spinless fermions, the *Heisenberg XYZ spin chain*, the *8 vertex model*; the other term is model dependent. The first term is dimensionally *marginal*, that is it tends to let the effective coupling constants grow logarithmically. But, if one could show that it is *exactly vanishing*, than the flow of the running coupling constants in all the above models could be controlled just by dimensional bounds, and the expansion would be convergent; the observables would then be expressed by explicit convergent series from which all the physical information can be extracted.

In the years two different strategies have been followed to prove the vanishing of the Beta function in the above sense. The first one, proposed by Benfatto and Gallavotti [BG1] and proved in [BoM][BM2], consists of an indirect argument, based on the fact that the first term of the Beta function (the one that is common to the class of models listed above) is the same as that one obtained from a multiscale analysis of the Luttinger model, that is an exactly solvable model [ML]; by contradiction, one shows that the Beta function must be vanishing, otherwise the correlation functions obtained by the multiscale integration would not coincide with the correlations which can be exactly computed from Luttinger's exact solution.

Very recently Benfatto and Mastropietro [BM1] proposed a new proof of the vanishing of the Beta function, completely independent from any exact solution and based on a rigorous implementation of Ward identities. Ward identities play a crucial role in Quantum Field Theory and Statistical Mechanics, as they allow to prove cancellations in a non perturbative way. The advantage of reducing the analysis of Ashkin-Teller to a fermionic model like that in (1.11) is that such model can be written as the sum of a term formally verifying many symmetries which were not verified by AT, *e.g.* *total gauge invariance* symmetry $\psi_{\mathbf{x},\omega}^\pm \rightarrow e^{\pm i\alpha_{\mathbf{x}}} \psi_{\mathbf{x},\omega}^\pm$ and *chiral gauge invariance* $\psi_{\mathbf{x},\omega}^\pm \rightarrow e^{\pm i\alpha_{\mathbf{x},\omega}} \psi_{\mathbf{x},\omega}^\pm$; plus mass terms and higher order corrections which are weighted by small constants. The first term has an associated beta function that is vanishing, as it can be

proved through Ward identities following from its gauge invariance⁶; the second term produces summable corrections to the Beta functions, which are specific of Ashkin–Teller. One says that the symmetries used to prove the vanishing of the Beta function are *hidden* in the spin models, as they are not verified even at a formal level; however they are exactly realized by a model that is “close”, in an RG sense, to Ashkin–Teller.

So, we use the argument of [BM1] together with a detailed analysis of the structure of the perturbative expansion to prove that λ_h stays small under the multiscale integration. Once this is established, we show that σ_h, μ_h, Z_h , under the iterations, evolve as: $\sigma_h \simeq \sigma_1 \gamma^{b_2 \lambda h}$, $\mu_h \simeq \mu_1 \gamma^{-b_2 \lambda h}$, $Z_h \simeq \gamma^{-b_1 \lambda^2 h}$, with b_1, b_2 explicitly computable in terms of a convergent power series.

We then perform the iterative integration described above up to a scale h_1^* such that $(|\sigma_{h_1^*}| + |\mu_{h_1^*}|) \gamma^{-h_1^*} = O(1)$. For scales lower than h_1^* we return to the description in terms of the original Majorana fermions $\psi^{(1, \leq h_1^*)}, \psi^{(2, \leq h_1^*)}$ associated with the two Ising subsystems. One of the two fields (say $\psi^{(1, \leq h_1^*)}$) is massive on scale h_1^* (so that the Ising subsystem with $j = 1$ is “far from criticality” on the same scale); then we can integrate the massive Majorana field $\psi^{(1, \leq h_1^*)}$ without any further multiscale analysis, obtaining an effective theory of a single Majorana field with mass $|\sigma_{h_1^*}| - |\mu_{h_1^*}|$, which can be arbitrarily small; this is equivalent to say that on scale h_1^* we have an effective description of the system as a single perturbed Ising model with *anomalous* parameters near criticality. The integration of the scales $\leq h_1^*$ is performed again by a multiscale decomposition similar to the one just described; an important feature is however that there are no more quartic marginal terms, because the anticommutativity of Grassmann variables forbids local quartic monomials of a single Majorana fermion. This greatly simplifies the analysis of the flow of the effective coupling constants, which is convergent, as it follows just by dimensional estimates. Criticality is found when the effective mass on scale $-\infty$ is vanishing; the values of t, u for which this happens are found by solving a non trivial implicit function problem.

Technically it is an interesting feature of this problem that there are two regimes in which the system must be described in terms of different fields: a first one in which the natural variables are Dirac Grassmann variables, and a second one in which they are Majorana; the scale h_1^* separating the two regimes is dynamically generated by the iterations. In the first regime the two entangled Ising subsystems are undistinguishable, the natural description is in terms of Dirac variables and the effective interaction is marginal; in the integration of such scales nonuniversal indexes appear and hidden Ward identities must be used to control the flow of the effective coupling constants. In the second region the two Ising subsystems really look different, one appears to be (almost) at criticality and the other far from criticality on the same scale; the parameters of the two subsystems are deeply changed (in an anomalous way) by the previous integration; in this region the effective interaction is irrelevant.

1.4. Summary.

In Chap.2 we get the exact solution of Ising by rewriting the partition function as a Grassmann functional integral. This will be the starting point for the subsequent perturbative construction.

In Chap.3 we describe the Grassmann formulation of a class of interacting Ising models in two dimensions, to which the multiscale method we will subsequently describe applies. This class includes the Ashkin–Teller model and the 8V model (and models obtained as perturbations of both). The general Grassmann formulation we describe is studied in detail for the Ashkin–Teller model and for the latter we also give an

⁶ At the formal level the proof of the vanishing of the Beta function through Ward identities is well-known since the 1970’s [DL][DM]. However the original proof of this statement discarded in the analysis the presence of cutoffs, which necessarily break exact gauge invariance; the problem of establishing whether gauge invariance and formal Ward identities were recovered in the limit of cutoff removal was not considered by the authors of the original proof. In [BM1] the authors first considered this problem and they proved that actually the Ward identities found after the removal of the cutoff are *different* from the formal ones: this is the phenomenon of chiral anomaly, well-known in the context of similar models used in Relativistic Quantum Field Theory, e.g. the Schwinger model [ZJ].

alternative Grassmann formulation that will be convenient in the following.

In Chap.4 we describe how to integrate out the ultraviolet degrees of freedom and we compute the effective action for the infrared part of the problem. We also study in detail the symmetry properties of our model, and we classify the terms that can possibly appear in the theory by symmetry reasons.

In Chap.5 we describe the multiscale analysis in the first regime of scales, where the system is described in terms of Dirac fields. In particular, this Chapter includes the definition of localization and a detailed analysis of the dimensional improvements that must be used to control the size of some contributions that, even if apparently marginal, can be shown to be effectively irrelevant.

In Chap.6 we study the flow of the running coupling constants, using the bounds previously derived in Chap. 5 and the vanishing of the Beta function.

In Chap.7 we describe the multiscale analysis in the second regime of scales, where the system is described in terms of a single Majorana field. We solve the equation for the scale h_1^* dividing the first and the second regime and the equation for the critical temperatures.

In Chap.8 we describe the expansion for the energy–energy correlation functions and we complete the proof of the main Theorem.

In the remaining Appendices we collect a number of technical lemmas needed for the proof of the main Theorem. In particular in Appendix A6 we reproduce the proof of the vanishing of the Beta function, following [BM1].

So, let's start.

2. The Ising model exact solution.

In this section we want to describe the Ising model exact solution, in a way that will be convenient for the subsequent perturbative analysis of the Ashkin–Teller model. We shall mainly follow the work of Samuel [S].

The Ising model partition function on a square lattice $\Lambda_M \subset \mathbb{Z}^2$, where M is the side of the square, is defined as:

$$\Xi_I = \sum_{\sigma_{\Lambda_M}} e^{\beta J \sum_{\langle i,j \rangle} \sigma_i \sigma_j}, \quad (2.1)$$

where $\langle i, j \rangle$ are nearest neighbor sites and β is the inverse temperature. We first consider open boundary conditions and, after that, the more complicated case of periodic boundary conditions.

2.1. The multipolygon representation.

It is well known that the partition function (2.1) is equivalent to the partition function of a gas of multipolygons with hard core. This representation was originally introduced to study the geometry of the microscopic configurations in the hot phase, and can be obtained as follows.

One first rewrite the sum appearing at the exponent in (2.1) as $\sum_b \tilde{\sigma}_b$, where \sum_b is the sum over the bonds linking nearest neighbor sites of Λ_M and $\tilde{\sigma}_b$ is the product of the spin variables over the two extremes of b . If we expand the exponential in power series we find:

$$\Xi_I = \sum_{\sigma_{\Lambda_M}} \prod_b (\cosh \beta J + \tilde{\sigma}_b \sinh \beta J) = (\cosh \beta J)^B \sum_{\sigma_{\Lambda_M}} \prod_b (1 + \tilde{\sigma}_b \tanh \beta J) \quad (2.2)$$

where B is the number of bonds of Λ_M . Developing the product, we are led to a sum of terms of the type:

$$(\tanh \beta J)^k \tilde{\sigma}_{b_1} \cdots \tilde{\sigma}_{b_k} \quad (2.3)$$

and we can conveniently describe them through the geometric set of lines b_1, \dots, b_k . If we perform the summation over the configurations σ_{Λ_M} , many terms of the form (2.3) give vanishing contribution. The only terms which survive are those in which the vertices of the geometric figure $b_1 \cup b_2 \cup \dots \cup b_k$ belong to an even number of b_j 's. These terms are those such that $\tilde{\sigma}_{b_1} \cdots \tilde{\sigma}_{b_k} \equiv 1$ and we shall call these geometric figures *multipolygons*. Let $P_k(\Lambda_M)$ be the number of multipolygons with k sides on the sublattice Λ_M . Then the partition function (2.1) is easily rewritten as:

$$\Xi_I = (\cosh \beta J)^B 2^{M^2} \sum_{k \geq 0} P_k(\Lambda_M) (\tanh \beta J)^k. \quad (2.4)$$

If open boundary conditions are assumed, only multipolygons *not* winding up the lattice are allowed. In the case of periodic boundary conditions the representation is the same, but the polygons are allowed to wind up the lattice.

2.2. The Grassmann integration rules.

In this section we introduce some basic definitions about Grassmann integration. We will need them to reinterpret (2.4) as a Grassmann functional integral.

Let us consider a finite dimensional *Grassman algebra*, which is a set of anticommuting *Grassman variables* $\{\psi_\alpha^+, \psi_\alpha^-\}$, with α an index belonging to some finite set A . This means that

$$\{\psi_\alpha^\sigma, \psi_{\alpha'}^{\sigma'}\} \equiv \psi_\alpha^\sigma \psi_{\alpha'}^{\sigma'} + \psi_{\alpha'}^{\sigma'} \psi_\alpha^\sigma = 0, \quad \forall \alpha, \alpha' \in A, \quad \forall \sigma, \sigma' = \pm; \quad (2.5)$$

in particular $(\psi_\alpha^\sigma)^2 = 0 \ \forall \alpha \in A$ and $\forall \sigma = \pm$.

Let us introduce another set of Grassman variables $\{d\psi_\alpha^+, d\psi_\alpha^-\}$, $\alpha \in A$, anticommuting with $\psi_\alpha^+, \psi_\alpha^-$, and an operation (*Grassman integration*) defined by

$$\int \psi_\alpha^\sigma d\psi_\alpha^\sigma = 1, \quad \int d\psi_\alpha^\sigma = 0, \quad a \in A, \quad \sigma = \pm 1. \quad (2.6)$$

If $F(\psi)$ is a polynomial in $\psi_\alpha^+, \psi_\alpha^-$, $\alpha \in A$, the operation

$$\int \prod_{\alpha \in A} d\psi_\alpha^+ d\psi_\alpha^- F(\psi) \quad (2.7)$$

is simply defined by iteratively applying (2.6) and taking into account the anticommutation rules (2.5). It is easy to check that for all $\alpha \in A$ and $C \in \mathbb{C}$

$$\frac{\int d\psi_\alpha^+ d\psi_\alpha^- e^{-\psi_\alpha^+ C \psi_\alpha^-} \psi_\alpha^- \psi_\alpha^+}{\int d\psi_\alpha^+ d\psi_\alpha^- e^{-\psi_\alpha^+ C \psi_\alpha^-}} = C^{-1}; \quad (2.8)$$

in fact $e^{-\psi_\alpha^+ C \psi_\alpha^-} = 1 - \psi_\alpha^+ C \psi_\alpha^-$ and by (2.6)

$$\int d\psi_\alpha^+ d\psi_\alpha^- e^{-\psi_\alpha^+ C \psi_\alpha^-} = C, \quad (2.9)$$

while

$$\int d\psi_\alpha^+ d\psi_\alpha^- e^{-\psi_\alpha^+ C \psi_\alpha^-} \psi_\alpha^- \psi_\alpha^+ = 1. \quad (2.10)$$

If one considers Grassmann variables whose quadratic action is not diagonal, one finds the generalizations of the above formulas, *e.g.*

$$\frac{\int \prod_{\alpha \in A} d\psi_\alpha^+ d\psi_\alpha^- e^{-\sum_{i,j \in A} \psi_i^+ M_{ij} \psi_j^-} \psi_{\alpha'}^- \psi_{\beta'}^+}{\int \prod_{\alpha \in A} d\psi_\alpha^+ d\psi_\alpha^- e^{-\sum_{i,j \in A} \psi_i^+ M_{ij} \psi_j^-}} = [M^{-1}]_{\alpha' \beta'}, \quad (2.11)$$

with M an $|A| \times |A|$ complex matrix. Again (2.11) can be easily verified by using (2.6) and the anticommutation rules (2.5), which also allow us to write

$$\int \prod_{\alpha \in A} d\psi_\alpha^+ d\psi_\alpha^- e^{-\sum_{i,j \in A} \psi_i^+ M_{ij} \psi_j^-} \equiv \det M \quad (2.12)$$

and

$$\int \prod_{\alpha \in A} d\psi_\alpha^+ d\psi_\alpha^- e^{-\sum_{i,j \in A} \psi_i^+ M_{ij} \psi_j^-} \psi_{\alpha'}^- \psi_{\beta'}^+ = M'_{\alpha' \beta'}, \quad (2.13)$$

if $M'_{\alpha' \beta'}$ is the minor complementary to the entry $M_{\alpha' \beta'}$.

The above formulae closely remind us the Gaussian integrals: note however that there is no need that M is real or positive defined (but of course they have to be invertible).

For the moment this is all we need for the Grassmann formulation of the Ising model. More algebraic properties of the Grassmann integration can be found in Appendix A1.

2.3. The Grassmann representation of the 2d Ising model with open boundary conditions.

In order to represent the sum over multipolygons in (2.4) as a Grassmann integral, we first associate to each site $\mathbf{x} \in \Lambda_M$, a set of four Grassmann variables, $\overline{H}_{\mathbf{x}}, H_{\mathbf{x}}, \overline{V}_{\mathbf{x}}, V_{\mathbf{x}}$, that must be thought as associated to four

FIG. 3. The four Grassmann fields associated to the sites \mathbf{x} and \mathbf{y} .

new sites drawn very near to \mathbf{x} and to its right, left, up side, down side respectively, see Fig 3. We shall denote these sites by $R_{\mathbf{x}}, L_{\mathbf{x}}, U_{\mathbf{x}}, D_{\mathbf{x}}$ respectively.

If $t \stackrel{\text{def}}{=} \tanh \beta J$, we consider the action

$$S(t) = t \sum_{\mathbf{x} \in \Lambda_M} [\overline{H}_{\mathbf{x}} H_{\mathbf{x}+\hat{e}_1} + \overline{V}_{\mathbf{x}} V_{\mathbf{x}+\hat{e}_0}] + \sum_{\mathbf{x} \in \Lambda_M} [\overline{H}_{\mathbf{x}} H_{\mathbf{x}} + \overline{V}_{\mathbf{x}} V_{\mathbf{x}} + \overline{V}_{\mathbf{x}} \overline{H}_{\mathbf{x}} + V_{\mathbf{x}} \overline{H}_{\mathbf{x}} + H_{\mathbf{x}} \overline{V}_{\mathbf{x}} + V_{\mathbf{x}} H_{\mathbf{x}}] , \quad (2.14)$$

where \hat{e}_1, \hat{e}_0 are the coordinate versors in the horizontal and vertical directions, respectively. Open boundary conditions are assumed. We claim that the following identity holds:

$$\frac{\Xi_I}{2^{M^2} (\cosh \beta J)^B} = (-1)^{M^2} \int \prod_{\mathbf{x} \in \Lambda_M} d\overline{H}_{\mathbf{x}} dH_{\mathbf{x}} d\overline{V}_{\mathbf{x}} dV_{\mathbf{x}} e^{S(t)} \quad (2.15)$$

where Ξ_I in the l.h.s. is calculated using open boundary conditions. The proof of (2.15) will occupy the rest of this section.

In order to prove (2.15) we expand the exponential in the r.h.s., we integrate term by term the Grassmann variables, and we get a summation over terms that we want to put in correspondence with the terms in the summation over multipolygons of (2.4). We can do as follows. We represent every quadratic term in (2.14) with a line connecting the two sites corresponding to the two Grassmann fields. Correspondingly, we represent every term obtained by the contraction of the Grassmann variables (that is the contraction of a suitable product of the quadratic terms appearing in $S(t)$) with the union of the lines representing the contracted monomials. The figure one obtains (call it a dimer) resembles a multipolygon, and exactly coincide with a multipolygon if one shrinks the sites $R_{\mathbf{x}}, L_{\mathbf{x}}, U_{\mathbf{x}}, D_{\mathbf{x}}$ to let them coincide with \mathbf{x} .

This graphical construction allows to put in correspondence each dimer with a unique multipolygon. We then have to show that the total weight of the dimer corresponding to the same multipolygon γ is exactly $(-1)^{M^2} t^{|\gamma|}$, where $(-1)^{M^2}$ is the same factor appearing in the r.h.s. of (2.15) (note that M^2 is the number of sites of Λ_M) and, if $|\gamma|$ is the length of γ , $t^{|\gamma|}$ is the weight (2.4) assigns to γ .

We first note that the correspondence between dimers and multipolygons is not one to one, because an empty site \mathbf{x} in the multipolygon representation corresponds to three different contractions of Grassmann fields, that is either to $\int d\overline{H}_{\mathbf{x}} dH_{\mathbf{x}} d\overline{V}_{\mathbf{x}} dV_{\mathbf{x}} \overline{H}_{\mathbf{x}} H_{\mathbf{x}} \overline{V}_{\mathbf{x}} V_{\mathbf{x}}$, or to $\int d\overline{H}_{\mathbf{x}} dH_{\mathbf{x}} d\overline{V}_{\mathbf{x}} dV_{\mathbf{x}} V_{\mathbf{x}} \overline{H}_{\mathbf{x}} H_{\mathbf{x}} \overline{V}_{\mathbf{x}}$, or to $\int d\overline{H}_{\mathbf{x}} dH_{\mathbf{x}} d\overline{V}_{\mathbf{x}} dV_{\mathbf{x}} V_{\mathbf{x}} H_{\mathbf{x}} \overline{V}_{\mathbf{x}} \overline{H}_{\mathbf{x}}$. The total contribution of these three contractions is:

$$\int d\overline{H}_{\mathbf{x}} dH_{\mathbf{x}} d\overline{V}_{\mathbf{x}} dV_{\mathbf{x}} (\overline{H}_{\mathbf{x}} H_{\mathbf{x}} \overline{V}_{\mathbf{x}} V_{\mathbf{x}} + V_{\mathbf{x}} \overline{H}_{\mathbf{x}} H_{\mathbf{x}} \overline{V}_{\mathbf{x}} + V_{\mathbf{x}} H_{\mathbf{x}} \overline{V}_{\mathbf{x}} \overline{H}_{\mathbf{x}}) = 1 - 1 - 1 = -1 , \quad (2.16)$$

as wanted.

It is easy to realize that, unless for the above ambiguity, the correspondence between dimers and multipolygons is unique. And, since each side of a dimer is weighted by a factor t and each empty site is weighted

by (-1) , the weights of the corresponding figures are the same, at least *in absolute value*. From now on we shall extract from the weight of γ the contribution of the empty sites together with the trivial factor $t^{|\gamma|}$ (that is we redefine the weight of γ by dividing it by $(-1)^{M^2 - n_\gamma t^{|\gamma|}}$, where n_γ is the number of sites belonging to γ , possibly different from $|\gamma|$, if γ has self intersections).

We are then left with proving that the weight of a dimer (as just redefined) is exactly $(-1)^{n_\gamma}$; in this way the sign of every configuration of dimers together with the minus signs of the empty sites, (2.16), would reproduce exactly the factor $(-1)^{M^2}$ in (2.15).

We start with considering the simplest dimer, that is the square with unit side. Let us denote its corner sites with $(0, 0) \equiv \mathbf{x}_1$, $(1, 0) \equiv \mathbf{x}_2$, $(1, 1) \equiv \mathbf{x}_3$, $(0, 1) \equiv \mathbf{x}_4$ and let us prove that its weight is $(-1)^4 = 1$. The explicit expression of its weight in terms of Grassmann integrals, as generated by the expansion of the exponent in (2.15) is:

$$\int \prod_{i=1}^4 d\overline{H}_{\mathbf{x}_i} dH_{\mathbf{x}_i} d\overline{V}_{\mathbf{x}_i} dV_{\mathbf{x}_i} \cdot \left[\overline{H}_{\mathbf{x}_1} H_{\mathbf{x}_2} \cdot V_{\mathbf{x}_2} \overline{H}_{\mathbf{x}_2} \cdot \overline{V}_{\mathbf{x}_2} V_{\mathbf{x}_3} \cdot \overline{V}_{\mathbf{x}_3} \overline{H}_{\mathbf{x}_3} \cdot (-H_{\mathbf{x}_3} \overline{H}_{\mathbf{x}_4}) \cdot H_{\mathbf{x}_4} \overline{V}_{\mathbf{x}_4} \cdot (-V_{\mathbf{x}_4} \overline{V}_{\mathbf{x}_1}) \cdot V_{\mathbf{x}_1} H_{\mathbf{x}_1} \right] \quad (2.17)$$

In the previous equation, we wrote the different binomials corresponding to the segments of the dimer following the anticlockwise order, starting from $\overline{H}_{\mathbf{x}_1}$. We associated a sign to each binomial, $+$ if its fields are written in the same order as they appear in (2.14), and $-$ otherwise.

By collecting the minus signs and by permutating the position of $\overline{H}_{\mathbf{x}_1}$ from the first to the last position, we find that (2.17) is equal to

$$- \int \prod_{i=1}^4 d\overline{H}_{\mathbf{x}_i} dH_{\mathbf{x}_i} d\overline{V}_{\mathbf{x}_i} dV_{\mathbf{x}_i} \cdot \left[H_{\mathbf{x}_2} V_{\mathbf{x}_2} \overline{H}_{\mathbf{x}_2} \overline{V}_{\mathbf{x}_2} \cdot V_{\mathbf{x}_3} \overline{V}_{\mathbf{x}_3} \overline{H}_{\mathbf{x}_3} H_{\mathbf{x}_3} \cdot \overline{H}_{\mathbf{x}_4} H_{\mathbf{x}_4} \overline{V}_{\mathbf{x}_4} V_{\mathbf{x}_4} \cdot \overline{V}_{\mathbf{x}_1} V_{\mathbf{x}_1} H_{\mathbf{x}_1} \overline{H}_{\mathbf{x}_1} \right] \quad (2.18)$$

where now we wrote separated from a dot the contributions corresponding to the same site. The explicit computation of (2.18) gives $-[(-1)(-1)(+1)(-1)] = +1$, as desired.

Let us now consider a generic dimer γ not winding up the lattice and without self intersections, and let us prove by induction that its weight is $(-1)^{n_\gamma}$. We will then assume that the dimers with number of sites $k \leq n_\gamma$ have weights $(-1)^k$. The first step from which the induction starts is the case $k = 4$, that we have just considered.

Let us consider the smallest rectangle R containing γ . Necessarily, each side of R has non empty intersection with γ . Let us enumerate the corners of γ which are also extremes of straight segments belonging to the sides of R , starting from the leftmost among the lowest of these points (possibly coinciding with the lower left corner of R) and proceeding in anticlockwise order; call \mathbf{x}_j the site with label j . Note that two consecutive indices $j, j+1$ could represent the same site $\mathbf{x}_j \equiv \mathbf{x}_{j+1} \in \Lambda_M$; in that case \mathbf{x} would be a corner of R . Call $2N$ the cardinality of the set of the enumerated points (it is even by construction) and let us identify the label $2N+1$ with the label 1.

Let us denote with the symbol $(2j-1 \rightarrow 2j)$, $j = 1, \dots, N$, the product of Grassmann fields corresponding to the straight line connecting the point $2j-1$ with $2j$ (not including the fields located in $2j-1$ and in $2j$), written in the anticlockwise order and with the sign induced by the expansion of the exponential in (2.15). That is, if the two fields belonging to a binomial appearing in (2.14), written following the anticlockwise order, are in the same order as they appear in (2.14), we will assign a $+$ sign to the second of those two fields (of course, second w.r.t. the anticlockwise order); otherwise a $-$ sign. As an example, if $2j-1$ and $2j$ are two points on the upper horizontal side of R , $(2j-1 \rightarrow 2j)$ would be equal to

$$(-\overline{H}_{\mathbf{x}_{2j-1}-\hat{e}_1}) \overline{V}_{\mathbf{x}_{2j-1}-\hat{e}_1} V_{\mathbf{x}_{2j-1}-\hat{e}_1} H_{\mathbf{x}_{2j-1}-\hat{e}_1} \cdots \cdots (-\overline{H}_{\mathbf{x}_{2j}+\hat{e}_1}) \overline{V}_{\mathbf{x}_{2j}+\hat{e}_1} V_{\mathbf{x}_{2j}+\hat{e}_1} H_{\mathbf{x}_{2j}+\hat{e}_1} \quad (2.19)$$

With a small abuse of notation, in the following we shall also denote with the symbol $(2j-1 \rightarrow 2j)$ the straight line connecting $2j-1$ with $2j$ on the polygon (*i.e.* the geometric object, not only the algebraic one).

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Moreover, let us denote with the symbol $[2j \rightarrow 2j+1]$, $j = 1, \dots, N$, the product of Grassmann fields corresponding to the *non* straight line connecting the point $2j$ with $2j+1$ (including the fields located in $2j$ and in $2j+1$) in the order induced by the choice of proceeding in anticlockwise order and with the sign induced by the expansion of the exponential in (2.15). With a small abuse of notation we shall also denote with the same symbol $[2j \rightarrow 2j+1]$ the corresponding line connecting $2j$ with $2j+1$ on the polygon γ . The sites $2j$ and $2j+1$ could either coincide (in that case $2j$ is a corner of R) or, if they do not, they could belong to the same side of R or to different adjacent sides of R . Let us denote with γ_j the union of $[2j \rightarrow 2j+1]$ with the shortest path on R connecting $2j$ with $2j+1$. The key remark is that $n_{\gamma_j} < n_\gamma$ so that, by the inductive hypothesis, the weight of γ_j is $(-1)^{n_{\gamma_j}}$.

With these notations and remarks, let us calculate the weight of γ . We write the weight in terms of a Grassmann integral as follows:

$$- \int \prod_{\mathbf{x} \in \gamma} d\overline{H}_{\mathbf{x}} dH_{\mathbf{x}} d\overline{V}_{\mathbf{x}} dV_{\mathbf{x}} \quad (1 \rightarrow 2)[2 \rightarrow 3] \cdots (2N-1 \rightarrow 2N)[2N \rightarrow 1] \quad (2.20)$$

The minus sign in front of the integral, appearing for the same reason why it appears in (2.18), is due to the permutation of the field $\overline{H}_{\mathbf{x}_1}$ from the first position (that is the one one gets by expanding the exponential in (2.15), writing the Grassmann binomials starting from site 1 and proceeding in anticlockwise order) to the last one (that is the position it appears into the product $[2N \rightarrow 1]$).

By a simple explicit calculation, it is straightforward to verify that the integral of the “straight line” $(2j-1 \rightarrow 2j)$ gives a contribution $(-1)^{\ell_{2j-1}-1}$, where ℓ_{2j-1} is the length of the segment $(2j-1 \rightarrow 2j)$ (note that $\ell_{2j-1} - 1$ is the number of sites belonging to $(2j-1 \rightarrow 2j)$, excluding the extremes). We are left with computing the integral of the “non straight line” $[2j \rightarrow 2j+1]$. We must distinguish 12 different cases, which we shall now study in detail.

1) j and $j+1$ are distinct and they belong to the low side of R . In this case

$$[j \rightarrow j+1] = \int H_{\mathbf{x}_j} \cdot V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot \{\overline{V}_{\mathbf{x}_j} \cdots (-\overline{V}_{\mathbf{x}_{j+1}})\} \cdot V_{\mathbf{x}_{j+1}} H_{\mathbf{x}_{j+1}} \cdot \overline{H}_{\mathbf{x}_{j+1}}, \quad (2.21)$$

as it follows from the rules explained above. We did not explicitly write neither the integration elements (those appearing in the r.h.s. of (2.15)) nor the fields corresponding to the sites between the site \mathbf{x}_j and the site \mathbf{x}_{j+1} ; note however that the number of fields between braces is necessarily even. In order to compute (2.21) we use the inductive hypothesis, telling us that the weight of γ_j is $(-1)^{n_{\gamma_j}}$, that is, explicitly:

$$(-1)^{D_j+d_j} = \int V_{\mathbf{x}_j} H_{\mathbf{x}_j} \cdot \overline{H}_{\mathbf{x}_j} (j \rightarrow j+1) H_{\mathbf{x}_{j+1}} \cdot V_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \cdot \{\overline{V}_{\mathbf{x}_j} \cdots (-\overline{V}_{\mathbf{x}_{j+1}})\}$$

In the last equation we called D_j the length of the non straight line $[j \rightarrow j+1]$ (note that $D_j + 1$ is the number of sites belonging to $[j \rightarrow j+1]$, including both extremes), we denoted by the symbol $(j \rightarrow j+1)$ the product of Grassmanian fields corresponding to the straight line on R connecting \mathbf{x}_j with \mathbf{x}_{j+1} and by d_j its length (note that $d_j - 1$ is the number of sites belonging to $(j \rightarrow j+1)$, excluding both extremes). By performing the integration over the fields in $(j \rightarrow j+1)$, we find:

$$\begin{aligned} (-1)^{D_j+1} &= \int V_{\mathbf{x}_j} H_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} H_{\mathbf{x}_{j+1}} V_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \{\overline{V}_{\mathbf{x}_j} \cdots (-\overline{V}_{\mathbf{x}_{j+1}})\} = \\ &= \int V_{\mathbf{x}_j} H_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \{\overline{V}_{\mathbf{x}_j} \cdots (-\overline{V}_{\mathbf{x}_{j+1}})\} H_{\mathbf{x}_{j+1}} V_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \end{aligned}$$

and the last line is clearly equal to the r.h.s. of (2.21).

2) j and $j+1$ coincide with the low right corner of R . In this case

$$[j \rightarrow j+1] = \int H_{\mathbf{x}_j} \cdot V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot \overline{V}_{\mathbf{x}_j} = -1. \quad (2.22)$$

3) j and $j+1$ are distinct and they belong to the low and the rights sides of R , respectively. In this case

$$[j \rightarrow j+1] = \int H_{\mathbf{x}_j} \cdot V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot \{\overline{V}_{\mathbf{x}_j} \cdots H_{\mathbf{x}_{j+1}}\} \cdot V_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \cdot \overline{V}_{\mathbf{x}_{j+1}}. \quad (2.23)$$

Calling $\mathbf{0}$ the lower right corner of R , the inductive hypothesis tells us that:

$$(-1)^{D_j+d_j} = \int V_{\mathbf{x}_j} H_{\mathbf{x}_j} \cdot \overline{H}_{\mathbf{x}_j} (j \rightarrow \mathbf{0}) H_{\mathbf{0}} \cdot V_{\mathbf{0}} \overline{H}_{\mathbf{0}} \cdot \overline{V}_{\mathbf{0}} (\mathbf{0} \rightarrow j+1) V_{\mathbf{x}_{j+1}} \cdot \overline{V}_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \cdot \{\overline{V}_{\mathbf{x}_j} \cdots H_{\mathbf{x}_{j+1}}\}.$$

In the last equation we called d_j the length of the shortest path on R connecting j with $j+1$ that is the sum of the lengths of $(j \rightarrow \mathbf{0})$ and $(\mathbf{0} \rightarrow j+1)$. By performing the integration over the fields in $(j \rightarrow \mathbf{0})$, in $\mathbf{0}$ and in $(\mathbf{0} \rightarrow j+1)$ we find:

$$\begin{aligned} (-1)^{D_j+1} &= \int V_{\mathbf{x}_j} H_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} V_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \{\overline{V}_{\mathbf{x}_j} \cdots H_{\mathbf{x}_{j+1}}\} = \\ &= \int V_{\mathbf{x}_j} H_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \{\overline{V}_{\mathbf{x}_j} \cdots H_{\mathbf{x}_{j+1}}\} V_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \end{aligned}$$

and the last line is clearly equal to the r.h.s. of (2.23).

4) j and $j+1$ are distinct and they belong to the right side of R . In this case

$$[j \rightarrow j+1] = \int V_{\mathbf{x}_j} \cdot \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot \{H_{\mathbf{x}_j} \cdots H_{\mathbf{x}_{j+1}}\} \cdot V_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \cdot \overline{V}_{\mathbf{x}_{j+1}}. \quad (2.24)$$

The inductive hypothesis tells us that:

$$(-1)^{D_j+d_j} = \int V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot \overline{V}_{\mathbf{x}_j} (j \rightarrow j+1) V_{\mathbf{x}_{j+1}} \cdot \overline{V}_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \cdot \{H_{\mathbf{x}_j} \cdots H_{\mathbf{x}_{j+1}}\}.$$

By performing the integration over the fields in $(j \rightarrow j+1)$ we find:

$$\begin{aligned} (-1)^{D_j+1} &= \int V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} V_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \{H_{\mathbf{x}_j} \cdots H_{\mathbf{x}_{j+1}}\} = \\ &= \int V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} \{H_{\mathbf{x}_j} \cdots H_{\mathbf{x}_{j+1}}\} V_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \end{aligned}$$

and the last line is clearly equal to the r.h.s. of (2.24).

5) j and $j+1$ coincide with the upper right corner of R . In this case

$$[j \rightarrow j+1] = \int V_{\mathbf{x}_j} \cdot \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot H_{\mathbf{x}_j} = -1. \quad (2.25)$$

6) j and $j+1$ are distinct and they belong to the right and upper sides of R , respectively. In this case

$$[j \rightarrow j+1] = \int V_{\mathbf{x}_j} \cdot \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot \{H_{\mathbf{x}_j} \cdots V_{\mathbf{x}_{j+1}}\} \cdot \overline{V}_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \cdot H_{\mathbf{x}_{j+1}}. \quad (2.26)$$

Calling $\mathbf{0}$ the upper right corner of R , the inductive hypothesis tells us that:

$$(-1)^{D_j+d_j} = \int V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot \overline{V}_{\mathbf{x}_j} (j \rightarrow \mathbf{0}) V_{\mathbf{0}} \cdot \overline{V}_{\mathbf{0}} \overline{H}_{\mathbf{0}} \cdot H_{\mathbf{0}} (\mathbf{0} \rightarrow j+1) (-\overline{H}_{\mathbf{x}_{j+1}}) \cdot H_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \cdot \{H_{\mathbf{x}_j} \cdots V_{\mathbf{x}_{j+1}}\}.$$

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By performing the integration over the fields in $(j \rightarrow \mathbf{0})$, in $\mathbf{0}$ and in $(\mathbf{0} \rightarrow j+1)$ we find:

$$\begin{aligned} (-1)^{D_j+1} &= \int V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} (-\overline{H}_{\mathbf{x}_{j+1}}) H_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \{H_{\mathbf{x}_j} \cdots V_{\mathbf{x}_{j+1}}\} = \\ &= \int V_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} \{H_{\mathbf{x}_j} \cdots V_{\mathbf{x}_{j+1}}\} (-\overline{H}_{\mathbf{x}_{j+1}}) H_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \end{aligned}$$

and the last line is clearly equal to the r.h.s. of (2.26).

7) j and $j+1$ are distinct and they belong to the upper side of R . In this case

$$[j \rightarrow j+1] = \int (-\overline{H}_{\mathbf{x}_j}) \cdot H_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} \cdot \{V_{\mathbf{x}_j} \cdots V_{\mathbf{x}_{j+1}}\} \cdot \overline{V}_{\mathbf{x}_{j+1}} \overline{H}_{\mathbf{x}_{j+1}} \cdot H_{\mathbf{x}_{j+1}} \cdot \quad (2.27)$$

The inductive hypothesis tells us that:

$$(-1)^{D_j+d_j} = \int \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot H_{\mathbf{x}_j} (j \rightarrow j+1) (-\overline{H}_{\mathbf{x}_{j+1}}) \cdot H_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \cdot \{V_{\mathbf{x}_j} \cdots V_{\mathbf{x}_{j+1}}\} \cdot$$

By performing the integration over the fields in $(j \rightarrow j+1)$ we find:

$$\begin{aligned} (-1)^{D_j+1} &= \int \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} H_{\mathbf{x}_j} (-\overline{H}_{\mathbf{x}_{j+1}}) H_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \{V_{\mathbf{x}_j} \cdots V_{\mathbf{x}_{j+1}}\} = \\ &= \int \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} H_{\mathbf{x}_j} \{V_{\mathbf{x}_j} \cdots V_{\mathbf{x}_{j+1}}\} (-\overline{H}_{\mathbf{x}_{j+1}}) H_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \end{aligned}$$

and the last line is clearly equal to the r.h.s. of (2.27).

8) j and $j+1$ coincide with the upper left corner of R . In this case

$$[j \rightarrow j+1] = \int (-\overline{H}_{\mathbf{x}_j}) \cdot H_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} \cdot V_{\mathbf{x}_j} = -1. \quad (2.28)$$

9) j and $j+1$ are distinct and they belong to the upper and left sides of R , respectively. In this case

$$[j \rightarrow j+1] = \int (-\overline{H}_{\mathbf{x}_j}) \cdot H_{\mathbf{x}_j} \overline{V}_{\mathbf{x}_j} \cdot \{V_{\mathbf{x}_j} \cdots (-\overline{H}_{\mathbf{x}_{j+1}})\} \cdot H_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \cdot V_{\mathbf{x}_{j+1}} \cdot \quad (2.29)$$

Calling $\mathbf{0}$ the upper left corner of R , the inductive hypothesis tells us that:

$$(-1)^{D_j+d_j} = \int \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} \cdot H_{\mathbf{x}_j} (j \rightarrow \mathbf{0}) (-\overline{H}_{\mathbf{0}}) \cdot H_{\mathbf{0}} \overline{V}_{\mathbf{0}} \cdot V_{\mathbf{0}} (\mathbf{0} \rightarrow j+1) (-\overline{V}_{\mathbf{x}_{j+1}}) \cdot V_{\mathbf{x}_{j+1}} H_{\mathbf{x}_{j+1}} \cdot \{V_{\mathbf{x}_j} \cdots (-\overline{H}_{\mathbf{x}_{j+1}})\} \cdot$$

By performing the integration over the fields in $(j \rightarrow \mathbf{0})$, in $\mathbf{0}$ and in $(\mathbf{0} \rightarrow j+1)$ we find:

$$\begin{aligned} (-1)^{D_j+1} &= \int \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} H_{\mathbf{x}_j} (-\overline{V}_{\mathbf{x}_{j+1}}) V_{\mathbf{x}_{j+1}} H_{\mathbf{x}_{j+1}} \{V_{\mathbf{x}_j} \cdots (-\overline{H}_{\mathbf{x}_{j+1}})\} = \\ &= \int \overline{V}_{\mathbf{x}_j} \overline{H}_{\mathbf{x}_j} H_{\mathbf{x}_j} \{V_{\mathbf{x}_j} \cdots (-\overline{H}_{\mathbf{x}_{j+1}})\} (-\overline{V}_{\mathbf{x}_{j+1}}) V_{\mathbf{x}_{j+1}} H_{\mathbf{x}_{j+1}} \end{aligned}$$

and the last line is clearly equal to the r.h.s. of (2.29).

10) j and $j+1$ are distinct and they belong to the left side of R . In this case

$$[j \rightarrow j+1] = \int (-\overline{V}_{\mathbf{x}_j}) \cdot V_{\mathbf{x}_j} H_{\mathbf{x}_j} \cdot \{\overline{H}_{\mathbf{x}_j} \cdots (-\overline{H}_{\mathbf{x}_{j+1}})\} \cdot H_{\mathbf{x}_{j+1}} \overline{V}_{\mathbf{x}_{j+1}} \cdot V_{\mathbf{x}_{j+1}} \cdot \quad (2.30)$$

The inductive hypothesis tells us that:

$$(-1)^{D_j+d_j} = \int H_{\mathbf{x}_j} \bar{V}_{\mathbf{x}_j} \cdot V_{\mathbf{x}_j} (j \rightarrow j+1) (-\bar{V}_{\mathbf{x}_{j+1}}) \cdot V_{\mathbf{x}_{j+1}} H_{\mathbf{x}_{j+1}} \cdot \{\bar{H}_{\mathbf{x}_j} \cdots (-\bar{H}_{\mathbf{x}_{j+1}})\}.$$

By performing the integration over the fields in $(j \rightarrow j+1)$ we find:

$$\begin{aligned} (-1)^{D_j+1} &= \int H_{\mathbf{x}_j} \bar{V}_{\mathbf{x}_j} V_{\mathbf{x}_j} (-\bar{V}_{\mathbf{x}_{j+1}}) V_{\mathbf{x}_{j+1}} H_{\mathbf{x}_{j+1}} \{\bar{H}_{\mathbf{x}_j} \cdots (-\bar{H}_{\mathbf{x}_{j+1}})\} = \\ &= \int H_{\mathbf{x}_j} \bar{V}_{\mathbf{x}_j} V_{\mathbf{x}_j} \{\bar{H}_{\mathbf{x}_j} \cdots (-\bar{H}_{\mathbf{x}_{j+1}})\} (-\bar{V}_{\mathbf{x}_{j+1}}) V_{\mathbf{x}_{j+1}} H_{\mathbf{x}_{j+1}} \end{aligned}$$

and the last line is clearly equal to the r.h.s. of (2.30).

11) j and $j+1$ coincide with the lower left corner of R . In this case it is necessarily $j \equiv 2N$ and $j+1 \equiv 1$ and we have:

$$[2N \rightarrow 1] = \int (-\bar{V}_{\mathbf{x}_1}) \cdot V_{\mathbf{x}_1} H_{\mathbf{x}_1} \cdot \bar{H}_{\mathbf{x}_1} = +1. \quad (2.31)$$

Note that this time the result is $+1$. This “wrong” sign exactly compensates the minus sign appearing in the r.h.s. of (2.20).

12) j and $j+1$ are distinct and they belong to the left and lower sides of R , respectively. In this case it is necessarily $j \equiv 2N$ and $j+1 \equiv 1$ and we have

$$[2N \rightarrow 1] = \int (-\bar{V}_{\mathbf{x}_{2N}}) \cdot V_{\mathbf{x}_{2N}} H_{\mathbf{x}_{2N}} \cdot \{\bar{H}_{\mathbf{x}_{2N}} \cdots (-\bar{V}_{\mathbf{x}_1})\} \cdot V_{\mathbf{x}_1} H_{\mathbf{x}_1} \cdot \bar{H}_{\mathbf{x}_1}. \quad (2.32)$$

Calling $\mathbf{0}$ the lower left corner of R , the inductive hypothesis tells us that:

$$(-1)^{D_N+d_N} = \int H_{\mathbf{x}_{2N}} \bar{V}_{\mathbf{x}_{2N}} \cdot V_{\mathbf{x}_{2N}} (2N \rightarrow \mathbf{0}) (-\bar{V}_{\mathbf{0}}) \cdot V_{\mathbf{0}} H_{\mathbf{0}} \cdot \bar{H}_{\mathbf{0}} (\mathbf{0} \rightarrow 1) H_{\mathbf{x}_1} \cdot V_{\mathbf{x}_1} \bar{H}_{\mathbf{x}_1} \cdot \{\bar{H}_{\mathbf{x}_{2N}} \cdots (-\bar{V}_{\mathbf{x}_1})\}.$$

By performing the integration over the fields in $(2N \rightarrow \mathbf{0})$, in $\mathbf{0}$ and in $(\mathbf{0} \rightarrow 1)$ we find:

$$\begin{aligned} (-1)^{D_N} &= \int H_{\mathbf{x}_{2N}} \bar{V}_{\mathbf{x}_{2N}} V_{\mathbf{x}_{2N}} H_{\mathbf{x}_1} V_{\mathbf{x}_1} \bar{H}_{\mathbf{x}_1} \{\bar{H}_{\mathbf{x}_{2N}} \cdots (-\bar{V}_{\mathbf{x}_1})\} = \\ &= \int H_{\mathbf{x}_{2N}} \bar{V}_{\mathbf{x}_{2N}} V_{\mathbf{x}_{2N}} \{\bar{H}_{\mathbf{x}_{2N}} \cdots (-\bar{V}_{\mathbf{x}_1})\} H_{\mathbf{x}_1} V_{\mathbf{x}_1} \bar{H}_{\mathbf{x}_1} \end{aligned}$$

and the last line is clearly equal to the r.h.s. of (2.29). It follows that $[2N \rightarrow 1] = -(-1)^{D_N+1}$, consistently with the result in item (11) above. Also in this case, the apparently “wrong” sign exactly compensates the minus sign appearing in the r.h.s. of (2.20).

Combining the results of previous items, we can simply say that the integration of $(2j-1 \rightarrow 2j)$ contributes to the weight of γ with $(-1)^{\ell_{2j-1}-1}$; the integration of $[2j \rightarrow 2j+1]$, with $j < N$, contributes with $(-1)^{L_{2j}+1}$ (here we defined L_{2j} to be the length of $[2j \rightarrow 2j+1]$), while $[2N \rightarrow 1]$ with $(-1)^{L_{2N}}$. Substituting these results into (2.20), we find that the weight of γ is equal to $(-1)^{n_\gamma}$, as desired.

The above discussion concludes the proof in the case of polygons without self intersections. Let us call *simple* a polygon without self intersections. If γ is not simple, calling ν_γ the number of its self intersections, we can easily prove that its weight is equal to $(-1)^{\nu_\gamma}$ times the product of the weights of a number of simple polygons, defined as follows. We draw with two colors, white and black, both the disconnected interiors

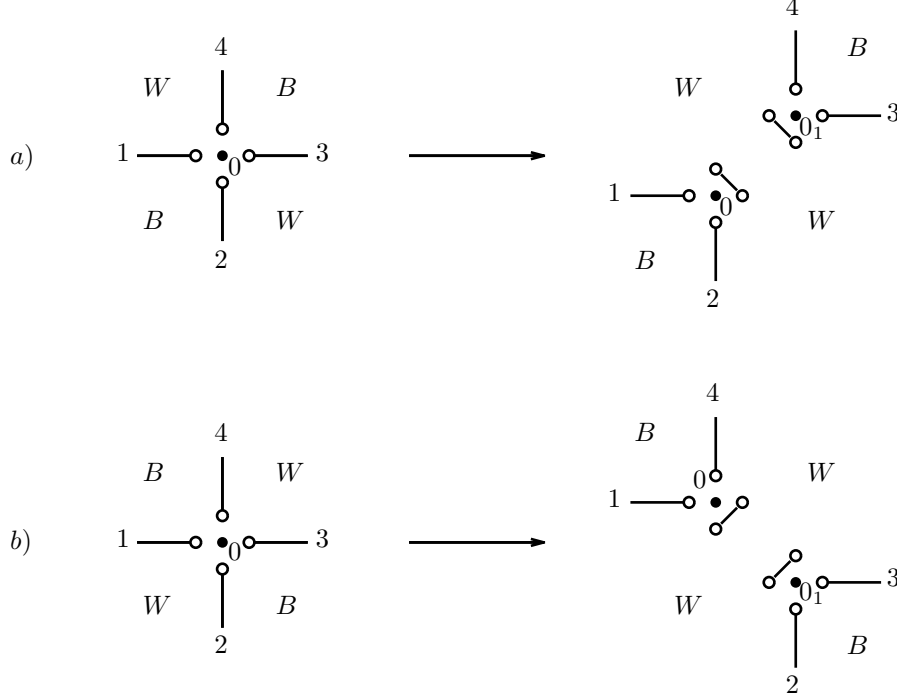


FIG. 4. The two elementary operations of disconnecting an intersection. The labels W and B mean that the corresponding regions must be coloured white and black respectively. Note that the operation of disconnecting an intersection involves the doubling of the site 0 at the center of the intersection: in the figure we call 0 and 0_1 its two copies after the disconnection.

of the polygon and its exterior, call them A_1, \dots, A_n and A_0 respectively. The drawing is done in such a way that A_0 is white and two adjacent sets A_i and A_j , $0 \leq i < j \leq n$, have different colors (we call A_i and A_j adjacent if their boundaries have a common side). Then we consider the set \mathcal{P} of simple polygons obtained as the boundaries of the black sets, thought as completely disconnected one from the other. The “disconnection” of the boundaries of the black regions (which originally could touch each other through the corners) is realized by the elementary disconnection of the intersection elements described in Fig.4.

We claim that the weight of γ is $(-1)^{\nu_\gamma} \prod_{\gamma' \in \mathcal{P}} (-1)^{n_{\gamma'}}$, which is the desired result (recall that \mathcal{P} is the set of polygons obtained as boundaries of the black sets, *after* the disconnection described in Fig.4). Note that the factor $(-1)^{\nu_\gamma}$ in front of the product of the weights of the disconnected simple polygons is due to the doubling of the centers of the intersections, implied by our definition of disconnection, see footnote to Fig. 4.

In order to prove the claim we explicitly write the contribution from the intersection in both cases (a) and (b) of Fig. 4, and we show that it is equal to the contribution of the two corner elements on the r.h.s. of Fig. 4, unless for a minus sign, to be associated to the new site 0_1 .

The contribution of the left hand side of case (a) in Fig. 4 is:

$$\int d\bar{H}_{\mathbf{x}_0} dH_{\mathbf{x}_0} d\bar{V}_{\mathbf{x}_0} dV_{\mathbf{x}_0} [\bar{H}_{\mathbf{x}_1} H_{\mathbf{x}_0} \cdot \bar{H}_{\mathbf{x}_0} H_{\mathbf{x}_3} \cdot \bar{V}_{\mathbf{x}_2} V_{\mathbf{x}_0} \cdot \bar{V}_{\mathbf{x}_0} V_{\mathbf{x}_4}] . \quad (2.33)$$

Multiplying (2.33) by

$$- \int d\bar{H}_{\mathbf{x}_{0_1}} dH_{\mathbf{x}_{0_1}} d\bar{V}_{\mathbf{x}_{0_1}} dV_{\mathbf{x}_{0_1}} [\bar{V}_{\mathbf{x}_{0_1}} \bar{H}_{\mathbf{x}_{0_1}} \cdot V_{\mathbf{x}_{0_1}} H_{\mathbf{x}_{0_1}}] = +1 , \quad (2.34)$$

we see that it can be equivalently rewritten as

$$\begin{aligned}
& - \int (\mathrm{d}\overline{H}_{\mathbf{x}_0} \mathrm{d}H_{\mathbf{x}_0} \mathrm{d}\overline{V}_{\mathbf{x}_0} \mathrm{d}V_{\mathbf{x}_0}) (\mathrm{d}\overline{H}_{\mathbf{x}_{01}} \mathrm{d}H_{\mathbf{x}_{01}} \mathrm{d}\overline{V}_{\mathbf{x}_{01}} \mathrm{d}V_{\mathbf{x}_{01}}) \cdot \\
& \cdot \left[\overline{H}_{\mathbf{x}_1} H_{\mathbf{x}_0} \cdot \overline{V}_{\mathbf{x}_{01}} \overline{H}_{\mathbf{x}_{01}} \cdot \overline{V}_{\mathbf{x}_2} V_{\mathbf{x}_0} \right] \cdot \left[\overline{H}_{\mathbf{x}_0} H_{\mathbf{x}_3} \cdot V_{\mathbf{x}_{01}} H_{\mathbf{x}_{01}} \cdot \overline{V}_{\mathbf{x}_0} V_{\mathbf{x}_4} \right] .
\end{aligned} \tag{2.35}$$

Exchanging the names of the fields $\overline{V}_{\mathbf{x}_0} \longleftrightarrow \overline{V}_{\mathbf{x}_{01}}$ and $\overline{H}_{\mathbf{x}_0} \longleftrightarrow \overline{H}_{\mathbf{x}_{01}}$, we easily recognize that (2.35) is equal to (-1) times the contribution of the r.h.s. of case (a) in Fig. 4. The minus sign compensates the fact that after the doubling the new polygon has a site more than the original one.

The argument can be repeated in case (b), so that the proof of the claim is complete.

This concludes the proof of (2.15) in the case of open boundary conditions (*i.e.* in the case where polygons winding up over the lattice are not allowed).

2.4. The Grassmann representation of the 2d Ising model with periodic boundary conditions.

In the case periodic boundary conditions are assumed, the representation in terms of multipolygons is the same, except for the fact that also polygons winding up over the lattice are allowed. In order to construct a Grassmann representation for the multipolygon expansion of Ising with p.b.c., let us start with considering the following expression:

$$\int \prod_{\mathbf{x} \in \Lambda_M} \mathrm{d}\overline{H}_{\mathbf{x}} \mathrm{d}H_{\mathbf{x}} \mathrm{d}\overline{V}_{\mathbf{x}} \mathrm{d}V_{\mathbf{x}} e^{S_{\varepsilon, \varepsilon'}(t)}, \tag{2.36}$$

where $\varepsilon, \varepsilon' = \pm$ and $S_{\varepsilon, \varepsilon'}(t)$ is defined by (2.14), but with different boundary conditions, *i.e.*

$$\begin{aligned}
\overline{H}_{\mathbf{x}+M\hat{e}_0} &= \varepsilon \overline{H}_{\mathbf{x}} \quad , \quad \overline{H}_{\mathbf{x}+M\hat{e}_1} = \varepsilon' \overline{H}_{\mathbf{x}} \\
H_{\mathbf{x}+M\hat{e}_0} &= \varepsilon H_{\mathbf{x}} \quad , \quad H_{\mathbf{x}+M\hat{e}_1} = \varepsilon' H_{\mathbf{x}}
\end{aligned} \quad , \quad \varepsilon, \varepsilon' = \pm, \tag{2.37}$$

where we recall that M is the side of the lattice Λ_M . Identical definitions are set for the variables V, \overline{V} . We shall say that $\overline{H}, H, \overline{V}, V$ satisfy ε -periodic (ε' -periodic) boundary conditions in vertical (horizontal) direction. Note that, unless for a sign and for the replacement $S(t) \rightarrow S_{\varepsilon, \varepsilon'}(t)$, (2.36) is the same as the r.h.s. of (2.15).

Clearly, by expanding the exponential in (2.36) and by integrating the Grassmann fields as described in previous section, we get a summation over dimers very similar to the one seen above. In particular the weights assigned to the closed polygons not winding up the lattice are exactly the same as those calculated in previous section. In this case, however, also Grassmann polygons winding up the lattice are allowed. Let us calculate the weight that (2.36) assigns to these polygons (as above we define the weight by discarding the “trivial” factors $t^{|\gamma|}$ and $(-1)^{M^2 - n_\gamma}$).

As an example, let us first calculate the contribution from the simplest polygon γ winding up the lattice, the horizontal straight line winding once in horizontal direction. Its weight is given by:

$$\int \overline{V}_0 V_0 \cdot \overline{H}_0 H_{\hat{e}_1} \cdot \overline{V}_{\hat{e}_1} V_{\hat{e}_1} \cdot \overline{H}_{\hat{e}_1} H_{2\hat{e}_1} \cdots \overline{H}_{(M-1)\hat{e}_1} H_{M\hat{e}_1} . \tag{2.38}$$

Now, using (2.37) we can rewrite $\overline{H}_{M\hat{e}_1}$ as $\varepsilon' H_0$. Also, permutating the field H_0 from the last position to the third one, we see that (2.38) is equal to:

$$\begin{aligned}
& (-\varepsilon') \int \overline{V}_0 V_0 H_0 \overline{H}_0 \cdot \overline{V}_{\hat{e}_1} V_{\hat{e}_1} H_{\hat{e}_1} \overline{H}_{\hat{e}_1} \cdots \overline{V}_{(M-1)\hat{e}_1} V_{(M-1)\hat{e}_1} H_{(M-1)\hat{e}_1} \overline{H}_{(M-1)\hat{e}_1} = \\
& = (-\varepsilon') (-1)^M = (-\varepsilon') (-1)^{n_\gamma} ,
\end{aligned} \tag{2.39}$$

where, in the last identity, we used that the length of the straight polygon γ is exactly M . Repeating the lengthy construction of previous section, it can be (straightforwardly) proven that a generic polygon γ winding up once in horizontal direction has a weight (as assigned by (2.36)) equal to $(-\varepsilon')(-1)^{n_\gamma}$. Analogously a polygon γ winding up once in horizontal direction has a weight (as assigned by (2.36)) equal to $(-\varepsilon)(-1)^{n_\gamma}$.

Let us now consider the simplest polygon γ winding up h times in horizontal direction and v times in vertical direction, that is the union of h distinct horizontal lines and v distinct vertical lines each of them winding once over the lattice in horizontal or vertical direction, respectively. Repeating the same simple calculation of (2.38)–(2.39), we easily see that the weight assigned by (2.36) to γ is $(-\varepsilon')^h(-\varepsilon)^v(-1)^{M(h+v)}$. Note that γ has $(-1)^{h \cdot v}$ self intersections, so that $n_\gamma = M(h+v) - h \cdot v$ and the weight can be rewritten as $(-\varepsilon')^h(-\varepsilon)^v(-1)^{h \cdot v}(-1)^{n_\gamma}$. Again, repeating the lengthy construction of previous section, it can be (straightforwardly) proven that a generic polygon γ winding up h times in horizontal direction and v times in vertical direction has a weight (as assigned by (2.36)) equal to $(-\varepsilon')^h(-\varepsilon)^v(-1)^{h \cdot v}(-1)^{n_\gamma}$.

Since the weight assigned to a generic polygon is the one just computed, which is in general different from $(-1)^{n_\gamma}$, it is clear that there exists *no choice* of $\varepsilon, \varepsilon' = \pm 1$ such that (2.36) is equal to $(-1)^{M^2} (2 \cosh^2 \beta J)^{-M^2}$ times Ξ_I , where now Ξ_I is the Ising model partition function in the volume Λ_M with periodic boundary conditions. However it is easy to realize that $(-1)^{M^2} \Xi_I (2 \cosh^2 \beta J)^{-M^2}$ is equal to a suitable linear combination of the expressions in (2.36), with different choices of $\varepsilon, \varepsilon' = \pm 1$: it holds that

$$(-1)^{M^2} \frac{\Xi_I}{(2 \cosh^2 \beta J)^{M^2}} = \frac{1}{2} \sum_{\varepsilon, \varepsilon' = \pm 1} \int \prod_{\mathbf{x} \in \Lambda_M} d\bar{H}_{\mathbf{x}} dH_{\mathbf{x}} d\bar{V}_{\mathbf{x}} dV_{\mathbf{x}} (-1)^{\delta_{(\varepsilon, \varepsilon')}} e^{S_{\varepsilon, \varepsilon'}(t)}, \quad (2.40)$$

where $\delta_{+, -} = \delta_{-, +} = \delta_{-, -} = 0$ and $\delta_{+, +} = 1$. In order to verify the last identity it is sufficient to verify that the weight assigned from the r.h.s. of (2.40) to each polygon γ is exactly $(-1)^{n_\gamma}$. If γ winds up the lattice h times in horizontal direction and v times in vertical direction, from the calculation above it follows that the weight assigned to γ by the r.h.s. of (2.40) is:

$$\begin{aligned} & \frac{1}{2} \sum_{\varepsilon, \varepsilon' = \pm 1} (-1)^{\delta_{(\varepsilon, \varepsilon')}} (-\varepsilon')^h (-\varepsilon)^v (-1)^{h \cdot v} (-1)^{n_\gamma} = \\ & = \frac{1}{2} (-1)^{n_\gamma} \left[(-1)^{h+v+hv+\delta_{+,+}} + (-1)^{v+hv+\delta_{+,-}} + (-1)^{h+hv+\delta_{-,+}} + (-1)^{hv+\delta_{-,-}} \right] \end{aligned} \quad (2.41)$$

The expression between square brackets on the last line is equal to $(-1)^{hv} [-(-1)^{h+v} + (-1)^v + (-1)^h + 1]$. Now, if h and v are both even, this is equal to $(+1)[-1 + 1 + 1 + 1] = 2$; if h is even and v is odd (or viceversa), it is equal to $(+1)[+1 - 1 + 1 + 1] = 2$; if they are both odd, it is equal to $(-1)[-1 - 1 - 1 + 1] = 2$. That is, (2.41) is identically equal to $(-1)^{n_\gamma}$, as wanted, and (2.40) is proven.

2.5. The Ising model's free energy

From the Grassmann representation of the Ising model partition function, it is easy to derive the well-known expression for the Ising's free energy. Even if in the following we won't need it, we reproduce here the calculation, for completeness.

The unitary transformation of the Grassmann fields diagonalizing the action $S_{\varepsilon, \varepsilon'}(t)$ is the following:

$$\begin{aligned} H_{\mathbf{x}} &= \frac{1}{|\Lambda_M|^{1/2}} \sum_{\mathbf{k} \in \mathcal{D}_{\varepsilon, \varepsilon'}} \hat{H}_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}}, & \bar{H}_{\mathbf{x}} &= \frac{1}{|\Lambda_M|^{1/2}} \sum_{\mathbf{k} \in \mathcal{D}_{\varepsilon, \varepsilon'}} \hat{\bar{H}}_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}}, \\ V_{\mathbf{x}} &= \frac{1}{|\Lambda_M|^{1/2}} \sum_{\mathbf{k} \in \mathcal{D}_{\varepsilon, \varepsilon'}} \hat{V}_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}}, & \bar{V}_{\mathbf{x}} &= \frac{1}{|\Lambda_M|^{1/2}} \sum_{\mathbf{k} \in \mathcal{D}_{\varepsilon, \varepsilon'}} \hat{\bar{V}}_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}}, \end{aligned} \quad (2.42)$$

where $\mathbf{k} = (k, k_0)$ and $\mathcal{D}_{\varepsilon, \varepsilon'}$ is the set of \mathbf{k} 's such that

$$k = \frac{2\pi n_1}{M} + \frac{(\varepsilon' - 1)\pi}{M} \quad k_0 = \frac{2\pi n_0}{M} + \frac{(\varepsilon - 1)\pi}{M} \quad (2.43)$$

with $-[M/2] \leq n_0 \leq [(M-1)/2]$, $-[M/2] \leq n_1 \leq [(M-1)/2]$, $n_0, n_1 \in \mathbb{Z}$. In terms of the new fields $\hat{H}_{\mathbf{k}}, \hat{\bar{H}}_{\mathbf{k}}, \hat{V}_{\mathbf{k}}, \hat{\bar{V}}_{\mathbf{k}}$, the action $S_{\varepsilon, \varepsilon'}(t)$ can be written as:

$$S_{\varepsilon, \varepsilon'}(t) = \sum_{\mathbf{k} \in \mathcal{D}_{\varepsilon, \varepsilon'}} \left[t \hat{\bar{H}}_{\mathbf{k}} \hat{H}_{-\mathbf{k}} e^{ik} + t \hat{\bar{V}}_{\mathbf{k}} \hat{V}_{-\mathbf{k}} e^{ik_0} + \hat{\bar{H}}_{\mathbf{k}} \hat{H}_{-\mathbf{k}} + \hat{\bar{V}}_{\mathbf{k}} \hat{V}_{-\mathbf{k}} + \hat{\bar{V}}_{\mathbf{k}} \hat{\bar{H}}_{-\mathbf{k}} + \hat{V}_{\mathbf{k}} \hat{\bar{H}}_{-\mathbf{k}} + \hat{H}_{\mathbf{k}} \hat{\bar{V}}_{-\mathbf{k}} + \hat{V}_{\mathbf{k}} \hat{H}_{-\mathbf{k}} \right] \quad (2.44)$$

Let us say that $\mathbf{k} > 0$ if its first component k_0 is > 0 . Then we can rewrite (2.44) as:

$$\begin{aligned} & \sum_{\mathbf{k} > 0} \left[t \hat{\bar{H}}_{\mathbf{k}} \hat{H}_{-\mathbf{k}} e^{ik} - t \hat{H}_{\mathbf{k}} \hat{\bar{H}}_{-\mathbf{k}} e^{-ik} + t \hat{\bar{V}}_{\mathbf{k}} \hat{V}_{-\mathbf{k}} e^{ik_0} - t \hat{V}_{\mathbf{k}} \hat{\bar{V}}_{-\mathbf{k}} e^{-ik_0} + \hat{\bar{H}}_{\mathbf{k}} \hat{H}_{-\mathbf{k}} - \hat{H}_{\mathbf{k}} \hat{\bar{H}}_{-\mathbf{k}} + \hat{\bar{V}}_{\mathbf{k}} \hat{V}_{-\mathbf{k}} - \hat{V}_{\mathbf{k}} \hat{\bar{V}}_{-\mathbf{k}} + \right. \\ & \quad \left. + \hat{\bar{V}}_{\mathbf{k}} \hat{\bar{H}}_{-\mathbf{k}} - \hat{\bar{H}}_{\mathbf{k}} \hat{\bar{V}}_{-\mathbf{k}} + \hat{V}_{\mathbf{k}} \hat{\bar{H}}_{-\mathbf{k}} - \hat{\bar{H}}_{\mathbf{k}} \hat{V}_{-\mathbf{k}} + \hat{H}_{\mathbf{k}} \hat{\bar{V}}_{-\mathbf{k}} - \hat{\bar{V}}_{\mathbf{k}} \hat{H}_{-\mathbf{k}} + \hat{V}_{\mathbf{k}} \hat{H}_{-\mathbf{k}} - \hat{H}_{\mathbf{k}} \hat{V}_{-\mathbf{k}} \right] \equiv \\ & \equiv \sum_{\mathbf{k} > 0} \Psi_{\mathbf{k}}^T M_{\mathbf{k}} \Psi_{-\mathbf{k}}, \end{aligned} \quad (2.45)$$

where $\Psi_{\mathbf{k}}^{T \text{def}} (\hat{\bar{H}}_{\mathbf{k}}, \hat{H}_{\mathbf{k}}, \hat{\bar{V}}_{\mathbf{k}}, \hat{V}_{\mathbf{k}})$ and the matrix $M_{\mathbf{k}}$ is defined as:

$$M_{\mathbf{k}}^{def} \begin{pmatrix} 0 & 1 + te^{ik} & -1 & -1 \\ -(1 + te^{-ik}) & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 + te^{ik_0} \\ 1 & 1 & -(1 + te^{-ik_0}) & 0 \end{pmatrix}. \quad (2.46)$$

Then, unless for a sign,

$$\int \prod_{\mathbf{x} \in \Lambda_M} d\bar{H}_{\mathbf{x}} dH_{\mathbf{x}} d\bar{V}_{\mathbf{x}} dV_{\mathbf{x}} e^{S_{\varepsilon, \varepsilon'}(t)} = \prod_{\mathbf{k} > 0} \left[\int d\hat{\bar{H}}_{\mathbf{k}} d\hat{\bar{H}}_{-\mathbf{k}} d\hat{H}_{\mathbf{k}} d\hat{H}_{-\mathbf{k}} d\hat{\bar{V}}_{\mathbf{k}} d\hat{\bar{V}}_{-\mathbf{k}} d\hat{V}_{\mathbf{k}} d\hat{V}_{-\mathbf{k}} \cdot e^{\Psi_{\mathbf{k}}^T M_{\mathbf{k}} \Psi_{-\mathbf{k}}} \right], \quad (2.47)$$

and, using (2.12), we see that the r.h.s. of (2.47) is equal $\prod_{\mathbf{k} > 0} \det M_{\mathbf{k}}$. The explicit computation of $\det M_{\mathbf{k}}$ leads to:

$$\begin{aligned} \det M_{\mathbf{k}} &= \left[1 + t^2 + 2t \cos k \right] \left[1 + t^2 + 2t \cos k_0 \right] - 4t(\cos k + \cos k_0) - 4t^2 \cos k \cos k_0 = \\ &= (1 + t^2)^2 - 2t(1 - t^2)(\cos k + \cos k_0). \end{aligned} \quad (2.48)$$

Now, using (2.40), we find that

$$\begin{aligned} & -\beta f_{Ising} \stackrel{def}{=} \lim_{M \rightarrow \infty} \frac{1}{M^2} \log \Xi_I = \\ &= \log(2 \cosh^2 \beta J) + \frac{1}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} \log \{ (1 + t^2)^2 - 2t(1 - t^2)(\cos k + \cos k_0) \} = \\ &= \frac{1}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} \log \left\{ 4 \left[\cosh^2 2\beta J - \sinh 2\beta J (\cos k + \cos k_0) \right] \right\}, \end{aligned} \quad (2.49)$$

that is the celebrated Onsager's result. Note that the argument of the logarithm in the last expression is always ≥ 0 and it vanishes iff $\sinh 2\beta J = 1$, that is the equation for the critical temperature. In the following we shall also write this condition in the equivalent form $\tanh \beta J = \sqrt{2} - 1$.

3. The Grassmann formulation of Ashkin–Teller.

In this section, using the Grassmann representation of the Ising model, derived in previous Chapter, we will derive the Grassmann representation for a class of interacting spin models, defined as a pair of Ising models coupled by suitable multi-spin interactions.

We will first derive the general representation for a wide class of models, to be defined in next section, including the Ashkin–Teller model defined in (1.1), the four states Potts model, the 8V model and the next to nearest neighbor Ising model. Then we will focus on AT, and we will perform more algebraic manipulations to get to a final representation that will be convenient for the following multiscale integration, necessary to construct a convergent expansion for some correlation functions, as explained in the Introduction.

The reason why we choose to focus on AT is for definiteness and for avoiding too cumbersome abstract expressions, that necessarily would turn out in trying to describe our method in a too general setting. However it will be clear that the same method we will apply to the study of the AT model could equally well be applied to 8V (in a suitable range of parameters), to Ising perturbed with a small non nearest neighbor interaction or to linear combinations of the above models. Note that, even if the four states Potts model can be represented by a Grassmann functional integral as proved below, the subsequent multiscale analysis we will apply to AT would *not* work for Potts. This is because Potts is equivalent to a system of strongly interacting fermions, while our perturbative methods are applicable only in the range of weak coupling.

3.1. The Grassmann representation for a pair of Ising models with multi spin interactions.

Let us start with considering a pair of nearest neighbor Ising models with periodic boundary conditions, labeled by $j = 1, 2$, with couplings allowed to depend on the bonds $b \in \Lambda_M^*$ (here Λ_M^* is the dual of Λ_M , that is the set of bonds linking the nearest neighbor sites of Λ_M):

$$H_I^{(j)} \{J_b^{(j)}\} = - \sum_{b \in \Lambda_M^*} J_b^{(j)} \tilde{\sigma}_b^{(j)}, \quad (3.1)$$

where the bond spin $\tilde{\sigma}_b^{(j)}$ was defined in §2.1 above. Repeating the construction of previous Chapter, one finds that the partition function of the model (3.1) can be written as:

$$\begin{aligned} \Xi_I^{(j)} &= \sum_{\sigma_{\Lambda_M}^{(j)}} e^{-\beta H_I^{(j)} \{J_b^{(j)}\}} = \\ &= (-1)^{M^2} 2^{M^2} \left[\prod_{b \in \Lambda_M^*} \cosh \beta J_b^{(j)} \right] \frac{1}{2} \sum_{\varepsilon, \varepsilon' = \pm} \int \prod_{\mathbf{x} \in \Lambda_M} dH_{\mathbf{x}}^{(j)} d\bar{H}_{\mathbf{x}}^{(j)} dV_{\mathbf{x}}^{(j)} d\bar{V}_{\mathbf{x}}^{(j)} (-1)^{\delta_\gamma} e^{S_\gamma^{(j)} \{t_b^{(j)}\}}, \end{aligned} \quad (3.2)$$

where $\gamma = (\varepsilon, \varepsilon')$ labels the boundary conditions of the Grassmann fields, δ_γ was defined after (2.40) and

$$\begin{aligned} S_\gamma^{(j)} \{t_b^{(j)}\} &= \sum_{\mathbf{x} \in \Lambda_M} \left[\tanh(J_{\mathbf{x}, \mathbf{x} + \hat{e}_1}^{(j)}) \bar{H}_{\mathbf{x}}^{(j)} H_{\mathbf{x} + \hat{e}_1}^{(j)} + \tanh(J_{\mathbf{x}, \mathbf{x} + \hat{e}_0}^{(j)}) \bar{V}_{\mathbf{x}}^{(j)} V_{\mathbf{x} + \hat{e}_0}^{(j)} \right] + \\ &+ \sum_{\mathbf{x} \in \Lambda_M} \left[\bar{H}_{\mathbf{x}}^{(j)} H_{\mathbf{x}}^{(j)} + \bar{V}_{\mathbf{x}}^{(j)} V_{\mathbf{x}}^{(j)} + \bar{V}_{\mathbf{x}}^{(j)} \bar{H}_{\mathbf{x}}^{(j)} + V_{\mathbf{x}}^{(j)} \bar{H}_{\mathbf{x}}^{(j)} + H_{\mathbf{x}}^{(j)} \bar{V}_{\mathbf{x}}^{(j)} + V_{\mathbf{x}}^{(j)} H_{\mathbf{x}}^{(j)} \right]. \end{aligned} \quad (3.3)$$

Let us now consider a multi spin interaction $V(\sigma^{(1)}, \sigma^{(2)})$ between the two layers, linear combination of interactions of the form:

$$\begin{aligned} V_I &= - \sum_{\mathbf{x} \in \Lambda_M} (\sigma_{\mathbf{x} + \mathbf{z}_1}^{(i_1)} \sigma_{\mathbf{x} + \mathbf{z}_1 + \hat{e}_{j_1}}^{(i_1)}) \cdot (\sigma_{\mathbf{x} + \mathbf{z}_2}^{(i_2)} \sigma_{\mathbf{x} + \mathbf{z}_2 + \hat{e}_{j_2}}^{(i_2)}) \cdots (\sigma_{\mathbf{x} + \mathbf{z}_k}^{(i_k)} \sigma_{\mathbf{x} + \mathbf{z}_k + \hat{e}_{j_k}}^{(i_k)}) \equiv \\ &\equiv - \sum_{\mathbf{x} \in \Lambda_M} \left[\prod_{(b, i) \in \mathcal{I}} \tilde{\sigma}_{b + \mathbf{x}}^{(i)} \right], \end{aligned} \quad (3.4)$$

where $\mathcal{I} = \{(b_p, i_p)\}_{p=1}^k$ is a set of indices (with $b_p \in \Lambda_M^*$ and $i_p = 1, 2$) and by $b + \mathbf{x}$ we denote the bond obtained by rigidly translating b of a vector \mathbf{x} . Periodic boundary conditions are assumed. Note that the interaction of Ashkin–Teller in (1.1) can be written as $\lambda V_{\mathcal{I}_1} + \lambda V_{\mathcal{I}_2}$, where, if we define b_0 to be the bond connecting $(0, 0)$ with $(0, 1)$ and b_1 that connecting $(0, 0)$ with $(1, 0)$, $\mathcal{I}_1 = \{(b_0, 1), (b_0, 2)\}$ and $\mathcal{I}_2 = \{(b_1, 1), (b_1, 2)\}$.

The key feature of the interaction (3.4) is to be a product of bond interactions appearing either in $H_I^{(1)}\{J_b^{(1)}\}$ or in $H_I^{(2)}\{J_b^{(2)}\}$, so that the partition function associated to the Hamiltonian $H_I^{(1)}\{J_b^{(1)}\} + H_I^{(2)}\{J_b^{(2)}\} + \lambda_1 V_{\mathcal{I}_1} + \dots + \lambda_n V_{\mathcal{I}_n}$ can be expressed a suitable derivative of $\Xi_I^{(1)} \Xi_I^{(2)}$ with respect to the couplings $J_b^{(j)}$. In fact:

$$\begin{aligned} \Xi &= \sum_{\sigma_{\Lambda_M}^{(1)}, \sigma_{\Lambda_M}^{(2)}} \exp \left\{ -\beta (H_I^{(1)}\{J_b^{(1)}\} + H_I^{(2)}\{J_b^{(2)}\} + \lambda_1 V_{\mathcal{I}_1} + \dots + \lambda_n V_{\mathcal{I}_n}) \right\} = \\ &= \sum_{\sigma_{\Lambda_M}^{(1)}, \sigma_{\Lambda_M}^{(2)}} \exp \left\{ \beta \sum_b (J_b^{(1)} \tilde{\sigma}_b^{(1)} + J_b^{(2)} \tilde{\sigma}_b^{(2)}) + \beta \sum_{q=1}^n \lambda_q \sum_{\mathbf{x}} \left[\prod_{(b,j) \in \mathcal{I}_q} \tilde{\sigma}_{b+\mathbf{x}}^{(j)} \right] \right\}. \end{aligned} \quad (3.5)$$

Defining $\hat{\lambda}_q \stackrel{def}{=} \tanh \beta \lambda_q$, the last expression can be rewritten as:

$$\begin{aligned} &\left[\prod_{q=1}^n \cosh \beta \lambda_q \right]^{M^2} \sum_{\sigma_{\Lambda_M}^{(1)}, \sigma_{\Lambda_M}^{(2)}} e^{\beta \sum_b (J_b^{(1)} \tilde{\sigma}_b^{(1)} + J_b^{(2)} \tilde{\sigma}_b^{(2)})} \prod_{\mathbf{x} \in \Lambda_M} \prod_{q=1}^n \left(1 + \hat{\lambda}_q \left[\prod_{(b,j) \in \mathcal{I}_q} \tilde{\sigma}_{b+\mathbf{x}}^{(j)} \right] \right) = \\ &= \left[\prod_{q=1}^n \cosh \beta \lambda_q \right]^{M^2} \prod_{\mathbf{x} \in \Lambda_M} \prod_{q=1}^n \left(1 + \hat{\lambda}_q \left[\prod_{(b,j) \in \mathcal{I}_q} \beta^{-1} \frac{\partial}{\partial J_{b+\mathbf{x}}^{(j)}} \right] \right) \Xi^{(1)}\{J_b^{(1)}\} \Xi^{(1)}\{J_b^{(1)}\}. \end{aligned} \quad (3.6)$$

Substituting into the r.h.s. of (3.6) the representation (3.2), we find that Ξ can be expressed as the sum of 16 Grassmann partition functions, differing for the boundary conditions and labeled by $\gamma_1 = (\varepsilon_1, \varepsilon'_1)$, $\gamma_2 = (\varepsilon_2, \varepsilon'_2)$:

$$\Xi = \left[4 \prod_{q=1}^n \cosh \beta \lambda_q \right]^{M^2} \frac{1}{4} \sum_{\gamma_1, \gamma_2} (-1)^{\delta_{\gamma_1} + \delta_{\gamma_2}} \Xi^{\gamma_1, \gamma_2}, \quad (3.7)$$

with Ξ^{γ_1, γ_2} given by:

$$\begin{aligned} &\prod_{\mathbf{x} \in \Lambda_M} \prod_{q=1}^n \left(1 + \hat{\lambda}_q \left[\prod_{(b,j) \in \mathcal{I}_q} \beta^{-1} \frac{\partial}{\partial J_{b+\mathbf{x}}^{(j)}} \right] \right) \cdot \left\{ \right. \\ &\cdot \left[\prod_{b \in \Lambda_M^*} \prod_{j=1}^2 \cosh \beta J_b^{(j)} \right] \int \left[\prod_{\mathbf{x} \in \Lambda_M} \prod_{j=1}^2 dH_{\mathbf{x}}^{(j)} d\bar{H}_{\mathbf{x}}^{(j)} dV_{\mathbf{x}}^{(j)} d\bar{V}_{\mathbf{x}}^{(j)} \right] e^{S_{\gamma}^{(1)}\{t_b^{(1)}\} + S_{\gamma}^{(2)}\{t_b^{(2)}\}} \left. \right\}. \end{aligned} \quad (3.8)$$

We now want to explicitly write the effect of the derivatives in the last expression and rewrite (3.8) as the Grassmann integral over an exponential of a (non quadratic) action. Let us first note that the effect of a singol derivative $\beta^{-1} \partial / \partial J_{b_0}^{(i)}$ over $[\prod_{b,j} \cosh \beta J_b^{(j)}] e^{S_{\gamma}^{(1)} + S_{\gamma}^{(2)}}$ is:

$$\beta^{-1} \frac{\partial}{\partial J_{b_0}^{(i)}} \left\{ \left[\prod_{b,j} \cosh \beta J_b^{(j)} \right] e^{S_{\gamma}^{(1)} + S_{\gamma}^{(2)}} \right\} = \left[\prod_{b,j} \cosh \beta J_b^{(j)} \right] e^{S_{\gamma}^{(1)} + S_{\gamma}^{(2)}} \left(t_{b_0}^{(j)} + s_{b_0}^{(j)} D_{b_0}^{(j)} \right), \quad (3.9)$$

where we introduced the definitions $t_b^{(j)} \stackrel{def}{=} \tanh \beta J_b^{(j)}$, $s_b^{(j)} \stackrel{def}{=} 1 / \cosh^2 \beta J_b^{(j)}$ and $D_b^{(j)}$ is a Grassmann binomial such that, if $b = (\mathbf{x}, \mathbf{x} + \hat{e}_0)$, $D_b^{(j)} \stackrel{def}{=} \bar{V}_{\mathbf{x}}^{(j)} V_{\mathbf{x}+\hat{e}_0}^{(j)}$ while, if $b = (\mathbf{x}, \mathbf{x} + \hat{e}_1)$, $D_b^{(j)} \stackrel{def}{=} \bar{H}_{\mathbf{x}}^{(j)} H_{\mathbf{x}+\hat{e}_1}^{(j)}$.

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Using (3.9) we see that (3.8) can be rewritten as:

$$\begin{aligned} & \left[\prod_{b \in \Lambda_M^*} \prod_{j=1}^2 \cosh \beta J_b^{(j)} \right] \int \left[\prod_{\mathbf{x} \in \Lambda_M} \prod_{j=1}^2 dH_{\mathbf{x}}^{(j)} d\bar{H}_{\mathbf{x}}^{(j)} dV_{\mathbf{x}}^{(j)} d\bar{V}_{\mathbf{x}}^{(j)} \right] \cdot \\ & \cdot e^{S_{\gamma}^{(1)}\{t_b^{(1)}\} + S_{\gamma}^{(2)}\{t_b^{(2)}\}} \prod_{\mathbf{x} \in \Lambda_M} \prod_{q=1}^n \left(1 + \hat{\lambda}_q \left[\prod_{(b,j) \in \mathcal{I}_q} \left(t_{b+\mathbf{x}}^{(j)} + s_{b+\mathbf{x}}^{(j)} D_{b+\mathbf{x}}^{(j)} \right) \right] \right). \end{aligned} \quad (3.10)$$

Let us denote with \mathbf{i} the elements of \mathcal{I}_q . and, if $\mathbf{i} = (b, j)$, define $\lambda_{\mathbf{x}}(\mathbf{i}) \stackrel{\text{def}}{=} s_{b+\mathbf{x}}^{(j)} / t_{b+\mathbf{x}}^{(j)}$ and $D_{\mathbf{x}}(\mathbf{i}) \stackrel{\text{def}}{=} D_{b+\mathbf{x}}^{(j)}$. Let us also assign an ordering to the elements of \mathcal{I} and let us write $\mathbf{i}_1 < \mathbf{i}_2$ if \mathbf{i}_1 precedes \mathbf{i}_2 w.r.t. this ordering. With these definitions we can rewrite the last product in (3.10) as:

$$\begin{aligned} & \prod_{\mathbf{x} \in \Lambda_M} \prod_{q=1}^n \left\{ 1 + \hat{\lambda}_q \left(\prod_{(b,j) \in \mathcal{I}_q} t_{b+\mathbf{x}}^{(j)} \right) \left[1 + \sum_{\mathbf{i}_1 \in \mathcal{I}_q} \lambda_{\mathbf{x}}(\mathbf{i}_1) D_{\mathbf{x}}(\mathbf{i}_1) + \sum_{\mathbf{i}_1 < \mathbf{i}_2} \lambda_{\mathbf{x}}(\mathbf{i}_1) D_{\mathbf{x}}(\mathbf{i}_1) \lambda_{\mathbf{x}}(\mathbf{i}_2) D_{\mathbf{x}}(\mathbf{i}_2) + \dots \right. \right. \\ & \left. \left. \dots + \sum_{\mathbf{i}_1 < \mathbf{i}_2 < \dots < \mathbf{i}_{|\mathcal{I}_q|}} \lambda_{\mathbf{x}}(\mathbf{i}_1) D_{\mathbf{x}}(\mathbf{i}_1) \dots \lambda_{\mathbf{x}}(\mathbf{i}_{|\mathcal{I}_q|}) D_{\mathbf{x}}(\mathbf{i}_{|\mathcal{I}_q|}) \right] \right\} \end{aligned} \quad (3.11)$$

and, calling $T_{\mathbf{x}}(\mathcal{I}_q) \stackrel{\text{def}}{=} \hat{\lambda}_q \left(\prod_{(b,j) \in \mathcal{I}_q} t_{b+\mathbf{x}}^{(j)} \right) \cdot \left[1 + \hat{\lambda}_q \left(\prod_{(b,j) \in \mathcal{I}_q} t_{b+\mathbf{x}}^{(j)} \right) \right]^{-1}$ we still can rewrite the last expression as:

$$\begin{aligned} & \prod_{\mathbf{x} \in \Lambda_M} \prod_{q=1}^n \left\{ \left(1 + \hat{\lambda}_q \prod_{(b,j) \in \mathcal{I}_q} t_{b+\mathbf{x}}^{(j)} \right) \cdot \left[1 + \sum_{\mathbf{i}_1 \in \mathcal{I}_q} T_{\mathbf{x}}(\mathcal{I}_q) \lambda_{\mathbf{x}}(\mathbf{i}_1) D_{\mathbf{x}}(\mathbf{i}_1) + \right. \right. \\ & \left. \left. + \sum_{\mathbf{i}_1 < \mathbf{i}_2} T_{\mathbf{x}}(\mathcal{I}_q) \lambda_{\mathbf{x}}(\mathbf{i}_1) D_{\mathbf{x}}(\mathbf{i}_1) \lambda_{\mathbf{x}}(\mathbf{i}_2) D_{\mathbf{x}}(\mathbf{i}_2) + \dots + \sum_{\mathbf{i}_1 < \mathbf{i}_2 < \dots < \mathbf{i}_{|\mathcal{I}_q|}} T_{\mathbf{x}}(\mathcal{I}_q) \lambda_{\mathbf{x}}(\mathbf{i}_1) D_{\mathbf{x}}(\mathbf{i}_1) \dots \lambda_{\mathbf{x}}(\mathbf{i}_{|\mathcal{I}_q|}) D_{\mathbf{x}}(\mathbf{i}_{|\mathcal{I}_q|}) \right] \right\} = \\ & = \prod_{\mathbf{x} \in \Lambda_M} \prod_{q=1}^n \left\{ \left(1 + \hat{\lambda}_q \prod_{(b,j) \in \mathcal{I}_q} t_{b+\mathbf{x}}^{(j)} \right) \exp \left\{ \sum_{\mathbf{i}_1 \in \mathcal{I}_q} \tilde{\lambda}_{\mathbf{x}}^{(q)}(\mathbf{i}_1) D_{\mathbf{x}}(\mathbf{i}_1) + \sum_{\mathbf{i}_1 < \mathbf{i}_2} \tilde{\lambda}_{\mathbf{x}}^{(q)}(\mathbf{i}_1, \mathbf{i}_2) D_{\mathbf{x}}(\mathbf{i}_1) D_{\mathbf{x}}(\mathbf{i}_2) + \dots + \right. \right. \\ & \left. \left. \dots + \sum_{\mathbf{i}_1 < \mathbf{i}_2 < \dots < \mathbf{i}_{|\mathcal{I}_q|}} \tilde{\lambda}_{\mathbf{x}}^{(q)}(\mathbf{i}_1, \dots, \mathbf{i}_{|\mathcal{I}_q|}) D_{\mathbf{x}}(\mathbf{i}_1) \dots D_{\mathbf{x}}(\mathbf{i}_{|\mathcal{I}_q|}) \right\} \right\}. \end{aligned} \quad (3.12)$$

In the last expression $\tilde{\lambda}_{\mathbf{x}}^{(q)}(\mathbf{i}_1) = T_{\mathbf{x}}(\mathcal{I}_q) \lambda_{\mathbf{x}}(\mathbf{i}_1)$ and the couplings $\tilde{\lambda}_{\mathbf{x}}^{(q)}(\mathbf{i}_1, \dots, \mathbf{i}_k)$, $2 \leq k \leq |\mathcal{I}_q|$, are defined by the following recursive relations:

$$T_{\mathbf{x}}(\mathcal{I}_q) \lambda_{\mathbf{x}}(\mathbf{i}_1) \dots \lambda_{\mathbf{x}}(\mathbf{i}_k) = \sum_{p=1}^k \sum_{\mathcal{J}_1 \cup \mathcal{J}_2 \cup \dots \cup \mathcal{J}_p = (\mathbf{i}_1, \dots, \mathbf{i}_k)} \tilde{\lambda}_{\mathbf{x}}^{(q)}(\mathcal{J}_1) \dots \tilde{\lambda}_{\mathbf{x}}^{(q)}(\mathcal{J}_p), \quad (3.13)$$

where $\mathcal{J}_r = (\mathbf{j}_1^{(r)}, \dots, \mathbf{j}_{|\mathcal{J}_r|}^{(r)})$ are ordered (i.e. $\mathbf{j}_1^{(r)} < \dots < \mathbf{j}_{|\mathcal{J}_r|}^{(r)}$) subsets of $(\mathbf{i}_1, \dots, \mathbf{i}_k)$, such that $|\mathcal{J}_1| + \dots + |\mathcal{J}_p| = k$.

Substituting (3.12) into (3.10) we finally find:

$$\begin{aligned} \Xi^{\gamma_1, \gamma_2} &= \left[\prod_{b \in \Lambda_M^*} \prod_{j=1}^2 \cosh \beta J_b^{(j)} \right] \left[\prod_{\mathbf{x} \in \Lambda_M} \prod_{q=1}^n \left(1 + \hat{\lambda}_q \prod_{(b,j) \in \mathcal{I}_q} t_{b+\mathbf{x}}^{(j)} \right) \right] \cdot \\ & \cdot \int \left[\prod_{\mathbf{x} \in \Lambda_M} \prod_{j=1}^2 dH_{\mathbf{x}}^{(j)} d\bar{H}_{\mathbf{x}}^{(j)} dV_{\mathbf{x}}^{(j)} d\bar{V}_{\mathbf{x}}^{(j)} \right] e^{S_{\gamma}^{(1)}\{t_b^{(1)}\} + S_{\gamma}^{(2)}\{t_b^{(2)}\} + V_{\lambda}}; \\ & V_{\lambda} \stackrel{\text{def}}{=} \sum_{\mathbf{x}} \sum_q \sum_{k=1}^{|\mathcal{I}_q|} \sum_{\mathbf{i}_1 < \mathbf{i}_2 < \dots < \mathbf{i}_k} \tilde{\lambda}_{\mathbf{x}}^{(q)}(\mathbf{i}_1, \dots, \mathbf{i}_k) D_{\mathbf{x}}(\mathbf{i}_1) \dots D_{\mathbf{x}}(\mathbf{i}_k). \end{aligned} \quad (3.14)$$

This concludes the derivation of the Grassmann representation for pairs of Ising models coupled by an interaction $V(\sigma^{(1)}, \sigma^{(2)})$, linear combination of interactions of the form (3.4).

3.2. The Grassmann representation for the Ashkin–Teller model.

We now specialize to the case of the Ashkin–Teller model. We first write the explicit form of V_λ in (3.14) for AT. As already discussed in previous section, the AT model corresponds to an interaction of the form $\lambda V_{\mathcal{I}_1} + \lambda V_{\mathcal{I}_2}$, with $\mathcal{I}_1 = \{(b_0, 1), (b_0, 2)\}$ and $\mathcal{I}_2 = \{(b_1, 1), (b_1, 2)\}$, b_0 being the bond connecting $(0, 0)$ with $(0, 1)$ and b_1 the one connecting $(0, 0)$ with $(1, 0)$. We shall assume the sets \mathcal{I}_q , $q = 1, 2$, ordered so that $(b_{q-1}, 1) < (b_{q-1}, 2)$. We are interested in writing the explicit expressions in the case $t_b^{(j)} \equiv t^{(j)}$ and $s_b^{(j)} \equiv s^{(j)}$ independent of b (but in general depending on the lattice $j = 1, 2$).

The definitions of λ_q , $T_{\mathbf{x}}$ and $\lambda_{\mathbf{x}}$ introduced in last section become in this case:

$$\hat{\lambda}_1 = \hat{\lambda}_2 = \tanh \beta \lambda \equiv \hat{\lambda}, \quad T_{\mathbf{x}}(\mathcal{I}_q) = \frac{\hat{\lambda} t^{(1)} t^{(2)}}{1 + \hat{\lambda} t^{(1)} t^{(2)}}, \quad \lambda_{\mathbf{x}}(b_{q-1}, j) = \frac{s^{(j)}}{t^{(j)}}. \quad (3.15)$$

Then (3.13) can be rewritten as:

$$\begin{aligned} \frac{\hat{\lambda} t^{(1)} t^{(2)}}{1 + \hat{\lambda} t^{(1)} t^{(2)}} \lambda_{\mathbf{x}}(b_{q-1}, 1) \lambda_{\mathbf{x}}(b_{q-1}, 2) &= \tilde{\lambda}_{\mathbf{x}}^{(q)} \left((b_{q-1}, 1), (b_{q-1}, 2) \right) + \tilde{\lambda}_{\mathbf{x}}(b_{q-1}, 1) \tilde{\lambda}_{\mathbf{x}}(b_{q-1}, 2), \\ \tilde{\lambda}_{\mathbf{x}}(b_{q-1}, j) &= \frac{\hat{\lambda} t^{(1)} t^{(2)}}{1 + \hat{\lambda} t^{(1)} t^{(2)}} \frac{s^{(j)}}{t^{(j)}}, \end{aligned} \quad (3.16)$$

implying:

$$\begin{aligned} \tilde{\lambda}_{\mathbf{x}}(b_{q-1}, 1) &= \frac{\hat{\lambda} s^{(1)} t^{(2)}}{1 + \hat{\lambda} t^{(1)} t^{(2)}} \equiv \lambda^{(1)}, \quad \tilde{\lambda}_{\mathbf{x}}(b_{q-1}, 2) = \frac{\hat{\lambda} s^{(2)} t^{(1)}}{1 + \hat{\lambda} t^{(1)} t^{(2)}} \equiv \lambda^{(2)}, \\ \tilde{\lambda}_{\mathbf{x}}^{(q)} \left((b_{q-1}, 1), (b_{q-1}, 2) \right) &= \frac{\hat{\lambda} s^{(1)} s^{(2)}}{(1 + \hat{\lambda} t^{(1)} t^{(2)})^2} \equiv \tilde{\lambda}. \end{aligned} \quad (3.17)$$

With these definitions V_λ in (3.14) can be written as:

$$\begin{aligned} V_\lambda = \sum_{\mathbf{x} \in \Lambda_M} \left\{ \left[\lambda^{(1)} \overline{H}_{\mathbf{x}}^{(1)} H_{\mathbf{x}+\hat{e}_1}^{(1)} + \lambda^{(2)} \overline{H}_{\mathbf{x}}^{(2)} H_{\mathbf{x}+\hat{e}_1}^{(2)} + \tilde{\lambda} \overline{H}_{\mathbf{x}}^{(1)} H_{\mathbf{x}+\hat{e}_1}^{(1)} \overline{H}_{\mathbf{x}}^{(2)} H_{\mathbf{x}+\hat{e}_1}^{(2)} \right] + \right. \\ \left. + \left[\lambda^{(1)} \overline{V}_{\mathbf{x}}^{(1)} V_{\mathbf{x}+\hat{e}_0}^{(1)} + \lambda^{(2)} \overline{V}_{\mathbf{x}}^{(2)} V_{\mathbf{x}+\hat{e}_0}^{(2)} + \tilde{\lambda} \overline{V}_{\mathbf{x}}^{(1)} V_{\mathbf{x}+\hat{e}_0}^{(1)} \overline{V}_{\mathbf{x}}^{(2)} V_{\mathbf{x}+\hat{e}_0}^{(2)} \right] \right\} \end{aligned} \quad (3.18)$$

and the first of (3.14) becomes:

$$\begin{aligned} \Xi_{AT}^{\gamma_1, \gamma_2} &= \left[(1 + \hat{\lambda} t^{(1)} t^{(2)}) \cosh \beta J^{(1)} \cosh \beta J^{(2)} \right]^{2M^2} \\ &\cdot \int \left[\prod_{\mathbf{x} \in \Lambda_M} \prod_{j=1}^2 dH_{\mathbf{x}}^{(j)} d\overline{H}_{\mathbf{x}}^{(j)} dV_{\mathbf{x}}^{(j)} d\overline{V}_{\mathbf{x}}^{(j)} \right] e^{S_\gamma^{(1)}(t_\lambda^{(1)}) + S_\gamma^{(2)}(t_\lambda^{(2)}) + \tilde{\lambda} \mathcal{V}}, \end{aligned} \quad (3.19)$$

where $S^{(j)}(t)$ was defined in previous Chapter, see (2.36); moreover $t_\lambda^{(j)} \stackrel{def}{=} t^{(j)} + \lambda^{(j)}$ and

$$\mathcal{V} = \sum_{\mathbf{x} \in \Lambda_M} \left(\overline{H}_{\mathbf{x}}^{(1)} H_{\mathbf{x}+\hat{e}_1}^{(1)} \overline{H}_{\mathbf{x}}^{(2)} H_{\mathbf{x}+\hat{e}_1}^{(2)} + \overline{V}_{\mathbf{x}}^{(1)} V_{\mathbf{x}+\hat{e}_0}^{(1)} \overline{V}_{\mathbf{x}}^{(2)} V_{\mathbf{x}+\hat{e}_0}^{(2)} \right). \quad (3.20)$$

3.3. Starting from (3.19) we will now make more algebraic manipulations by introducing new Grassmann fields, linear combinations of the fields $\overline{H}, H, \overline{V}, V$. This will be convenient in order to set up the Renormalization Group scheme we will use to study in detail the specific heat of AT. The aim is to rewrite the formal

action appearing at the exponent in (3.19) as the formal action of a perturbed *massive Luttinger model*, the latter being a model for which Renormalization Group technique is already well developed [BM][GM].

We shall consider for simplicity the partition function $\Xi_{AT}^- \stackrel{\text{def}}{=} \Xi_{AT}^{(-,-),(-,-)}$, *i.e.* the partition function in which all Grassmannian variables verify antiperiodic boundary conditions. The other fifteen partition functions in the analogue of (3.7) admit similar expressions. In fact we shall see in Chap. 7 and Appendix A9 that, if (λ, t, u) *does not belong* to the *critical surface*¹ the partition function $\Xi_{AT}^{\gamma_1, \gamma_2}$ divided by $\Xi_I^{(1)\gamma_1} \Xi_I^{(2)\gamma_2}$ is exponentially insensitive to boundary conditions as $M \rightarrow \infty$.

It is convenient to perform the following change of variables [ID], $j = 1, 2$

$$\begin{aligned} \overline{H}_{\lambda, \mathbf{x}}^{(j)} + iH_{\lambda, \mathbf{x}}^{(j)} &= e^{i\frac{\pi}{4}} \psi_{\mathbf{x}}^{(j)} - e^{i\frac{\pi}{4}} \chi_{\mathbf{x}}^{(j)} & \overline{H}_{\lambda, \mathbf{x}}^{(j)} - iH_{\lambda, \mathbf{x}}^{(j)} &= e^{-i\frac{\pi}{4}} \overline{\psi}_{\mathbf{x}}^{(j)} - e^{-i\frac{\pi}{4}} \overline{\chi}_{\mathbf{x}}^{(j)} \\ \overline{V}_{\lambda, \mathbf{x}}^{(j)} + iV_{\lambda, \mathbf{x}}^{(j)} &= \psi_{\mathbf{x}}^{(j)} + \chi_{\mathbf{x}}^{(j)} & \overline{V}_{\lambda, \mathbf{x}}^{(j)} - iV_{\lambda, \mathbf{x}}^{(j)} &= \overline{\psi}_{\mathbf{x}}^{(j)} + \overline{\chi}_{\mathbf{x}}^{(j)}. \end{aligned} \quad (3.21)$$

The effect of this change of variables is the following one. If $S^{(j)}(t_\lambda^{(j)}) \stackrel{\text{def}}{=} \sum_{\mathbf{x}} S_{\mathbf{x}}^{(j)}$, after the change of variables (2.21) we get:

$$S_{\mathbf{x}}^{(j)} = S_{\mathbf{x}}^{(j, \psi)} + S_{\mathbf{x}}^{(j, \chi)} + Q_{\mathbf{x}}^{(j)} \quad (3.22)$$

where

$$\begin{aligned} S_{\mathbf{x}}^{(j, \psi)} &= \frac{t_\lambda^{(j)}}{4} \left[\psi_{\mathbf{x}}^{(j)} (\partial_1 - i\partial_0) \psi_{\mathbf{x}}^{(j)} + \overline{\psi}_{\mathbf{x}}^{(j)} (\partial_1 + i\partial_0) \overline{\psi}_{\mathbf{x}}^{(j)} \right] + \\ &+ \frac{t_\lambda^{(j)}}{4} \left[-i\overline{\psi}_{\mathbf{x}}^{(j)} (\partial_1 \psi_{\mathbf{x}}^{(j)} + \partial_0 \psi_{\mathbf{x}}^{(j)}) + i\psi_{\mathbf{x}}^{(j)} (\partial_1 \overline{\psi}_{\mathbf{x}}^{(j)} + \partial_0 \overline{\psi}_{\mathbf{x}}^{(j)}) \right] + i \left(\sqrt{2} - 1 - t_\lambda^{(j)} \right) \overline{\psi}_{\mathbf{x}}^{(j)} \psi_{\mathbf{x}}^{(j)} \end{aligned} \quad (3.23)$$

with

$$\partial_1 \psi_{\mathbf{x}}^{(j)} = \psi_{\mathbf{x}+\hat{e}_1}^{(j)} - \psi_{\mathbf{x}}^{(j)} \quad \partial_0 \psi_{\mathbf{x}}^{(j)} = \psi_{\mathbf{x}+\hat{e}_0}^{(\alpha)} - \psi_{\mathbf{x}}^{(j)}. \quad (3.24)$$

Moreover

$$\begin{aligned} S_{\mathbf{x}}^{(j, \chi)} &= \frac{t_\lambda^{(j)}}{4} \left[\chi_{\mathbf{x}}^{(j)} (\partial_1 - i\partial_0) \chi_{\mathbf{x}}^{(j)} + \overline{\chi}_{\mathbf{x}}^{(j)} (\partial_1 + i\partial_0) \overline{\chi}_{\mathbf{x}}^{(j)} \right] + \\ &+ \frac{t_\lambda^{(j)}}{4} \left[-i\overline{\chi}_{\mathbf{x}}^{(j)} (\partial_1 \chi_{\mathbf{x}}^{(j)} + \partial_0 \chi_{\mathbf{x}}^{(j)}) + i\chi_{\mathbf{x}}^{(j)} (\partial_1 \overline{\chi}_{\mathbf{x}}^{(j)} + \partial_0 \overline{\chi}_{\mathbf{x}}^{(j)}) \right] - i \left(\sqrt{2} + 1 + t_\lambda^{(j)} \right) \overline{\chi}_{\mathbf{x}}^{(j)} \chi_{\mathbf{x}}^{(j)} \end{aligned} \quad (3.25)$$

and finally

$$\begin{aligned} Q_{\mathbf{x}}^{(j)} &= \frac{t_\lambda^{(j)}}{4} \left[-\psi_{\mathbf{x}}^{(j)} (\partial_1 \chi_{\mathbf{x}}^{(j)} + i\partial_0 \chi_{\mathbf{x}}^{(j)}) - \overline{\psi}_{\mathbf{x}}^{(j)} (\partial_1 \overline{\chi}_{\mathbf{x}}^{(j)} - i\partial_0 \overline{\chi}_{\mathbf{x}}^{(j)}) - \right. \\ &- \chi_{\mathbf{x}}^{(j)} (\partial_1 \psi_{\mathbf{x}}^{(j)} + i\partial_0 \psi_{\mathbf{x}}^{(j)}) - \overline{\chi}_{\mathbf{x}}^{(j)} (\partial_1 \overline{\psi}_{\mathbf{x}}^{(j)} - i\partial_0 \overline{\psi}_{\mathbf{x}}^{(j)}) + i\overline{\psi}_{\mathbf{x}}^{(j)} (\partial_1 \chi_{\mathbf{x}}^{(j)} - \partial_0 \chi_{\mathbf{x}}^{(j)}) + \\ &\left. + i\psi_{\mathbf{x}}^{(j)} (-\partial_1 \overline{\chi}_{\mathbf{x}}^{(j)} + \partial_0 \overline{\chi}_{\mathbf{x}}^{(j)}) + i\overline{\chi}_{\mathbf{x}}^{(j)} (\partial_1 \psi_{\mathbf{x}}^{(j)} - \partial_0 \psi_{\mathbf{x}}^{(j)}) + i\chi_{\mathbf{x}}^{(j)} (-\partial_1 \overline{\psi}_{\mathbf{x}}^{(j)} + \partial_0 \overline{\psi}_{\mathbf{x}}^{(j)}) \right]. \end{aligned} \quad (3.26)$$

Formally $S_{\mathbf{x}}^{j, \psi}$ and $S_{\mathbf{x}}^{j, \chi}$ are the actions of a pair of *Majorana* $d = 2$ fermions on a lattice with masses $\sqrt{2} - 1 - t_\lambda^{(j)}$, and $\sqrt{2} + 1 + t_\lambda^{(j)}$, respectively; note that, since $-c|\lambda| \leq t_\lambda^{(j)} \leq 1 + c|\lambda|$, for some $c > 0$, the mass of the χ field is always $O(1)$. On the contrary the mass of the ψ field can be arbitrarily small; in the free case ($\lambda = 0$) the condition for the theory to be massless is equivalent to the condition $t = \sqrt{2} - 1$, that

¹ The critical surface is a suitable 2-dimensional subset of $[-\varepsilon, \varepsilon] \times D \times [-\frac{|D|}{2}, \frac{|D|}{2}]$, that is of the 3-dim set in the parameters space where we are interested to study the AT model, see the assumptions in the main Theorem in the Introduction; we will explicitly determine the critical surface in Chap. 7 below and we will prove that it can be parametrized as $(\lambda, t_c^\pm(\lambda, u), u)$, with $t_c^\pm(\lambda, u)$ given by (1.7).

is the Ising's criticality condition (see the end of Chap.3); this is consistent with the well-known property that the Ising's correlation functions decay as power laws if and only if we are at criticality.

It is convenient to pass from Majorana to *Dirac* fermions via the change of variables

$$\psi_{1,\mathbf{x}}^\mp = \frac{1}{\sqrt{2}}(\psi_{\mathbf{x}}^{(1)} \pm i\psi_{\mathbf{x}}^{(2)}), \quad \psi_{-1,\mathbf{x}}^\mp = \frac{1}{\sqrt{2}}(\bar{\psi}_{\mathbf{x}}^{(1)} \pm i\bar{\psi}_{\mathbf{x}}^{(2)}), \quad (3.27)$$

$$\chi_{1,\mathbf{x}}^\mp = \frac{1}{\sqrt{2}}(\chi_{\mathbf{x}}^{(1)} \pm i\chi_{\mathbf{x}}^{(2)}), \quad \chi_{-1,\mathbf{x}}^\mp = \frac{1}{\sqrt{2}}(\bar{\chi}_{\mathbf{x}}^{(1)} \pm i\bar{\chi}_{\mathbf{x}}^{(2)}) \quad (3.28)$$

and, if $\alpha = \pm$, $\omega = \pm 1$, we define $\hat{\phi}_{\omega,\mathbf{k}}^\alpha \stackrel{def}{=} \sum_{\mathbf{x}} e^{-i\alpha\mathbf{k}\mathbf{x}} \phi_{\omega,\mathbf{x}}^\alpha$, with ϕ denoting either ψ or χ .

Let us introduce some more definitions. Let

$$t_\lambda \stackrel{def}{=} \frac{t_\lambda^{(1)} + t_\lambda^{(2)}}{2}, \quad u_\lambda \stackrel{def}{=} \frac{t_\lambda^{(1)} - t_\lambda^{(2)}}{2} \quad (3.29)$$

and note that t_λ, u_λ as functions of t, u are given by

$$t_\lambda = t \frac{1 + \hat{\lambda}}{1 + \hat{\lambda}(t^2 - u^2)}, \quad u_\lambda = u \frac{1 - \hat{\lambda}}{1 + \hat{\lambda}(t^2 - u^2)}. \quad (3.30)$$

Furthermore, let

$$Q(\psi, \chi) \stackrel{def}{=} \sum_{\mathbf{x}, j} Q_{\mathbf{x}}^{(j)}, \quad V(\psi, \chi) \stackrel{def}{=} \mathcal{V}, \quad (3.31)$$

where $Q(\psi, \chi)$ and $V(\psi, \chi)$ must be thought as functions of ψ^\pm and χ^\pm . With the above definitions and using (2.13), (2.28) it is straightforward algebra to verify that Ξ_{AT}^- can be rewritten as:

$$\Xi_{AT}^- = e^{-EM^2} \int P(d\psi) P(d\chi) e^{Q(\psi, \chi) + \bar{\lambda} V(\psi, \chi)}, \quad (3.32)$$

where E is a suitable constant (we won't need its explicit value) and $P(d\phi)$, $\phi = \psi, \chi$, is:

$$P(d\phi) = \mathcal{N}_\phi^{-1} \prod_{\mathbf{k} \in D_{-, -}} \prod_{\omega = \pm 1} d\phi_{\mathbf{k}, \omega}^+ d\phi_{\mathbf{k}, \omega}^- \exp \left\{ -\frac{t_\lambda}{4M^2} \sum_{\mathbf{k} \in D_{-, -}} \mathbf{\Phi}_{\mathbf{k}}^{+, T} A_\phi(\mathbf{k}) \mathbf{\Phi}_{\mathbf{k}} \right\},$$

$$A_\phi(\mathbf{k}) = \begin{pmatrix} i \sin k + \sin k_0 & -i\sigma_\phi(\mathbf{k}) & -\frac{\mu}{2}(i \sin k + \sin k_0) & i\mu(\mathbf{k}) \\ i\sigma_\phi(\mathbf{k}) & i \sin k - \sin k_0 & -i\mu(\mathbf{k}) & -\frac{\mu}{2}(i \sin k - \sin k_0) \\ -\frac{\mu}{2}(i \sin k + \sin k_0) & i\mu(\mathbf{k}) & i \sin k + \sin k_0 & -i\sigma_\phi(\mathbf{k}) \\ -i\mu(\mathbf{k}) & -\frac{\mu}{2}(i \sin k - \sin k_0) & i\sigma_\phi(\mathbf{k}) & i \sin k - \sin k_0 \end{pmatrix}, \quad (3.33)$$

where

$$\mathbf{\Phi}^{+, \mathbf{T}}_{\mathbf{k}} = (\hat{\phi}_{1,\mathbf{k}}^+, \hat{\phi}_{-1,\mathbf{k}}^+, \hat{\phi}_{1,-\mathbf{k}}^-, \hat{\phi}_{-1,-\mathbf{k}}^-), \quad \mathbf{\Phi}^{\mathbf{T}}_{\mathbf{k}} = (\hat{\phi}_{1,\mathbf{k}}^-, \hat{\phi}_{-1,\mathbf{k}}^-, \hat{\phi}_{1,-\mathbf{k}}^+, \hat{\phi}_{-1,-\mathbf{k}}^+), \quad (3.34)$$

\mathcal{N}_ϕ is chosen in such a way that $\int P(d\phi) = 1$ and

$$\sigma_\phi(\mathbf{k}) = 2 \left(1 + \frac{\pm \sqrt{2} + 1}{t_\lambda} \right) + \cos k_0 + \cos k - 2, \quad \mu(\mathbf{k}) = -(u_\lambda/t_\lambda)(\cos k + \cos k_0). \quad (3.35)$$

In the first of (2.37) the $- (+)$ sign corresponds to $\phi = \psi$ ($\phi = \chi$). The parameter μ in (3.33) is given by $\mu \stackrel{def}{=} \mu(\mathbf{0})$.

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It is convenient to split the $\sqrt{2} - 1$ appearing in the definition of $\sigma_\psi(\mathbf{k})$ as:

$$\sqrt{2} - 1 = (\sqrt{2} - 1 + \frac{\nu}{2}) - \frac{\nu}{2} \stackrel{\text{def}}{=} t_\psi - \frac{\nu}{2}, \quad (3.36)$$

where ν is a parameter to be properly chosen later as a function of λ , in such a way that the average location of the critical points will be given by $t_\lambda = t_\psi$; in other words ν has the role of a *counterterm* fixing the middle point of the critical temperatures. The splitting (3.36) induces the following splitting of $P(d\psi)$:

$$P(d\psi) = P_\sigma(d\psi)e^{-\nu F_\nu(\psi)} \quad , \quad F_\nu(\psi) \stackrel{\text{def}}{=} \frac{1}{2M^2} \sum_{\mathbf{k}, \omega} (-i\omega) \hat{\psi}_{\omega, \mathbf{k}}^+ \hat{\psi}_{-\omega, \mathbf{k}}^-, \quad (3.37)$$

where $P_\sigma(d\psi)$ is given by (2.29) with $\phi = \psi$ and $\sigma \stackrel{\text{def}}{=} 2(1 - t_\psi/t_\lambda)$ replacing $\sigma_\psi(\mathbf{0})$.

The final expression we found will be the starting point for the multiscale analysis of the partition function and of the correlation function, which will occupy us in the following Chapters.

4. The ultraviolet integration.

Starting from this Chapter and up to Chapter 7, we will construct the expansion for the free energy f of the Ashkin–Teller model, see (1.3), and we will prove that f is well defined and analytic in λ, t, u for any $t \neq t_c^\pm$, see (1.7).

It will soon be clear that a naive perturbative expansion in $\tilde{\lambda}$ of the Grassmann functional integral in (3.32) would give us poor bounds for the partition function. This is because the propagator of the ψ fields introduced in last Chapter has a mass that is *vanishing* at $t_\lambda = \sqrt{2} - 1 + \frac{\nu}{2} \pm u$, that is in correspondence of the “bare” critical points. This produces infrared divergences in the integrals defining the n -th order contribution to the free energy, as obtained by this naive perturbative expansion. It is then necessary to find out an iterative resummation rule, giving sense to the perturbation series. The iterative construction we will develop is inspired to the multiscale analysis of Grassmann functional integrals similar to (3.32), as those appearing in the context of non relativistic spinless fermions in 1+1 dimensions or of 1-dim quantum spin chains [BGPS][GM][BM]; in all these problems the partition function can be written as the integral of an exponential of a fermionic action, of the form of a Luttinger model action plus a perturbation, containing both a quadratic and a quartic term. In our case, looking at (3.32) and (3.33), the Luttinger model part of the action corresponds to the diagonal elements of $A_\phi(\mathbf{k})$ plus the *local part* of $\tilde{\lambda}V$; the quadratic corrections to the non diagonal terms of $A_\phi(\mathbf{k})$; the quartic corrections to the *non local part* of $\tilde{\lambda}V$. The difference between our problem and those already studied in the literature consists in the form of this perturbation; more precisely, in the form of the quadratic corrections, which can be *relevant* or *marginal* in a Renormalization Group sense, see next Chapter. These terms generate new *effective coupling constants*, whose size must be controlled throughout the Renormalization Group iterations. Moreover, our problem, formulated as a problem of 1-dim fermions, does not have many natural symmetries that usually are present in a fermionic theory, such as gauge symmetry, conservation of the particle number and of the quasi-particle number. A priori, this could be a reason why other relevant or marginal terms, not originally present in the action (3.32), could be generated by the iterative construction. We will use a number of hidden symmetries, induced by the symmetries of the original spin model, to guarantee that these terms are not generated; by “hidden” here we mean that these symmetries, very natural in the original spin language, are not apparent in the fermionic one.

In this Chapter we describe the integration of the ultraviolet degrees of freedom, that is of the massive fields in (3.32) (*i.e.* the χ fields). This will be the first step of our iterative construction. The subsequent steps will be for various aspects technically very similar to the ultraviolet one, which we will now present in all details. We will introduce and describe many of the technical tools we will use throughout the work, such as the Pfaffian expansion, the Gram–Hadamard bounds and the symmetry relations for the fermionic fields.

4.1. The effective interaction on scale 1.

The propagators $\langle \phi_{\mathbf{x},\omega}^\sigma \phi_{\mathbf{y},\omega'}^{\sigma'} \rangle$ of the fermionic integration $P(d\phi)$, defined in (3.33), verify the following bound, for some $A, \kappa > 0$:

$$|\langle \phi_{\mathbf{x},\omega}^\sigma \phi_{\mathbf{y},\omega'}^{\sigma'} \rangle| \leq A e^{-\kappa \bar{m}_\phi |\mathbf{x}-\mathbf{y}|}, \quad (4.1)$$

where \bar{m}_ϕ is the minimum between $|m_\phi^{(1)}|$ and $|m_\phi^{(2)}|$ and

$$m_\phi^{(1)} \stackrel{\text{def}}{=} 2(t_\lambda^{(1)} - t_\phi)/t_\lambda, \quad m_\phi^{(2)} \stackrel{\text{def}}{=} 2(t_\lambda^{(1)} - t_\phi)/t_\lambda, \quad (4.2)$$

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where $\phi = \psi, \chi$, t_ψ was defined in (3.36) and $t_\chi \stackrel{def}{=} -\sqrt{2} - 1$. Note that both $m_\chi^{(1)}$ and $m_\chi^{(2)}$ are $O(1)$. This suggests to integrate first the χ variables.

Aim of the present and of the subsequent sections is to perform the integration of the χ variables and, after that, to rewrite (3.32) in the form

$$\Xi_{AT}^- = e^{-M^2 E_1} \int P_{Z_1, \sigma_1, \mu_1, C_1}(d\psi) e^{-\mathcal{V}^{(1)}(\sqrt{Z_1} \psi)}, \quad \mathcal{V}^{(1)}(0) = 0, \quad (4.3)$$

where $C_1(\mathbf{k}) \equiv 1$, $Z_1 = t_\psi$, $\sigma_1 = \sigma/(1 - \frac{\sigma}{2})$, $\mu_1 = \mu/(1 - \frac{\sigma}{2})$ and $P_{Z_1, \sigma_1, \mu_1, C_1}(d\psi)$ is the exponential of a quadratic form:

$$\begin{aligned} P_{Z_1, \sigma_1, \mu_1, C_1}(d\psi) &= \mathcal{N}_1^{-1} \prod_{\mathbf{k} \in D_{-, -}}^{\omega=\pm 1} d\psi_{\omega, \mathbf{k}}^+ d\psi_{\omega, \mathbf{k}}^- \exp \left[-\frac{1}{4M^2} \sum_{\mathbf{k} \in D_{-, -}} Z_1 C_1(\mathbf{k}) \Psi_{\mathbf{k}}^{+, T} A_\psi^{(1)}(\mathbf{k}) \Psi_{\mathbf{k}} \right], \\ A_\psi^{(1)}(\mathbf{k}) &= \begin{pmatrix} M^{(1)}(\mathbf{k}) & N^{(1)}(\mathbf{k}) \\ N^{(1)}(\mathbf{k}) & M^{(1)}(\mathbf{k}) \end{pmatrix} \\ M^{(1)}(\mathbf{k}) &= \begin{pmatrix} i \sin k + \sin k_0 + a_1^+(\mathbf{k}) & -i(\sigma_1 + c_1(\mathbf{k})) \\ i(\sigma_1 + c_1(\mathbf{k})) & i \sin k - \sin k_0 + a_1^-(\mathbf{k}) \end{pmatrix} \\ N^{(1)}(\mathbf{k}) &= \begin{pmatrix} b_1^+(\mathbf{k}) & i(\mu_1 + d_1(\mathbf{k})) \\ -i(\mu_1 + d_1(\mathbf{k})) & b_1^-(\mathbf{k}) \end{pmatrix}, \end{aligned} \quad (4.4)$$

where \mathcal{N}_1 is chosen in such a way that $\int P_{Z_1, \sigma_1, \mu_1, C_1}(d\psi) = 1$. We shall call $\mathcal{V}^{(1)}$ the *effective interaction* on scale 1; it can be expressed as a sum of monomials in ψ of arbitrary order:

$$\mathcal{V}^{(1)}(\psi) = \sum_{n=1}^{\infty} \sum_{\mathbf{k}_1, \dots, \mathbf{k}_{2n}} \prod_{\substack{\underline{\alpha}, \underline{\omega} \\ i=1}}^{2n} \hat{\psi}_{\omega_i, \mathbf{k}_i}^{\alpha_i(\leq 1)} \widehat{W}_{2n, \underline{\alpha}, \underline{\omega}}^{(1)}(\mathbf{k}_1, \dots, \mathbf{k}_{2n-1}) \delta(\sum_{i=1}^{2n} \alpha_i \mathbf{k}_i) \quad (4.5)$$

where $\underline{\alpha} = (\alpha_1, \dots, \alpha_{2n})$, $\underline{\omega} = (\omega_1, \dots, \omega_{2n})$, $\alpha_i = \pm 1$, $\omega_i = \pm 1$ and $\delta(\mathbf{k}) = \sum_{\mathbf{n} \in \mathbb{Z}^2} \delta_{\mathbf{k}, 2\pi \mathbf{n}}$. The constant E_1 in (4.3), the functions $a_1^\pm, b_1^\pm, c_1, d_1$ in (4.4) and the kernels $\widehat{W}_{2n, \underline{\alpha}, \underline{\omega}}^{(1)}$ in (4.5) satisfy natural dimensional bounds and a number of symmetry relations, which will be described and proved below, where we will also show in detail how to get to (4.3). At the end of the Chapter we will collect the results in Theorem 4.1.

Note that from now on we will consider all functions appearing in the theory as functions of λ, σ_1, μ_1 (of course t and u can be analytically and elementarily expressed in terms of λ, σ_1, μ_1). We shall also assume $|\sigma_1|, |\mu_1|$ bounded by some $O(1)$ constant. Note that if $t \pm u$ belong to a sufficiently small interval D centered around $\sqrt{2} - 1$, as assumed in the hypothesis of the Main Theorem in the Introduction, then of course $|\sigma_1|, |\mu_1| \leq c_1$ for a suitable constant c_1 (for instance, if $D = [\frac{3(\sqrt{2}-1)}{4}, \frac{5(\sqrt{2}-1)}{4}]$, that is a possible choice for the interval D , we find $|\sigma_1| \leq 1 + O(\varepsilon)$ and $|\mu_1| \leq 2 + O(\varepsilon)$).

4.2. The integration of the χ fields.

We start with considering (3.32), with $P(d\psi)$ rewritten as in (3.37), and we define:

$$e^{-\tilde{E}_1 M^2 - Q^{(1)}(\psi) - \mathcal{V}^{(1)}(\psi)} \stackrel{def}{=} \int P(d\chi) e^{Q(\psi, \chi) - \nu F_\sigma(\psi) + \tilde{\lambda} V(\psi, \chi)}, \quad (4.6)$$

where \tilde{E}_1 is a constant, $Q^{(1)}$ is quadratic in ψ and $O(1)$ w.r.t. λ, ν and $\mathcal{V}^{(1)}$ is at least quadratic in ψ and $O(\lambda, \nu)$. $Q^{(1)}$ will contribute to the free measure $P_{Z_1, \sigma_1, \mu_1, C_1}$.

We calculate $\mathcal{V}^{(1)}$ in terms of *truncated expectations* (see Appendix A1), defined as:

$$\mathcal{E}_\chi^T(X; n) = \frac{\partial^n}{\partial \alpha^n} \log \int P(d\chi) e^{\alpha X(\chi)}|_{\alpha=0}, \quad (4.7)$$

where $P(d\chi)$ is defined in (3.33) and the associated propagator is given by

$$\begin{aligned}
g_{(-,\omega),(+,\omega)}^\chi(\mathbf{x}-\mathbf{y}) &\stackrel{def}{=} \langle \chi_{\mathbf{x},\omega}^- \chi_{\mathbf{y},\omega}^+ \rangle = g_\omega^{\chi(1)}(\mathbf{x}-\mathbf{y}) + g_\omega^{\chi(2)}(\mathbf{x}-\mathbf{y}) \\
g_{(-,\omega),(+,-\omega)}^\chi(\mathbf{x}-\mathbf{y}) &\stackrel{def}{=} \langle \chi_{\mathbf{x},\omega}^- \chi_{\mathbf{y},-\omega}^+ \rangle = g_{\omega,-\omega}^{\chi(1)}(\mathbf{x}-\mathbf{y}) + g_{\omega,-\omega}^{\chi(2)}(\mathbf{x}-\mathbf{y}) \\
g_{(\alpha,\omega),(\alpha,-\omega)}^\chi(\mathbf{x}-\mathbf{y}) &\stackrel{def}{=} \langle \chi_{\mathbf{x},\omega}^\alpha \chi_{\mathbf{y},-\omega}^\alpha \rangle = g_{\omega,-\omega}^{\chi(1)}(\mathbf{x}-\mathbf{y}) - g_{\omega,-\omega}^{\chi(2)}(\mathbf{x}-\mathbf{y}) \\
g_{(\alpha,\omega),(\alpha,\omega)}^\chi(\mathbf{x}-\mathbf{y}) &\stackrel{def}{=} \langle \chi_{\mathbf{x},\omega}^\alpha \chi_{\mathbf{y},\omega}^\alpha \rangle = g_\omega^{\chi(1)}(\mathbf{x}-\mathbf{y}) - g_\omega^{\chi(2)}(\mathbf{x}-\mathbf{y}) ,
\end{aligned} \tag{4.8}$$

where, for $j = 1, 2$,

$$\begin{aligned}
g_\omega^{\chi(j)}(\mathbf{x}-\mathbf{y}) &= \frac{2}{t_\lambda} \frac{1}{M^2} \sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \frac{\zeta_j (-i \sin k + \omega \sin k_0)}{\zeta_j^2 (\sin^2 k + \sin^2 k_0) + (m_{\chi,\mathbf{k}}^{(j)})^2} \\
g_{\omega,-\omega}^{\chi(j)}(\mathbf{x}-\mathbf{y}) &= \frac{2}{t_\lambda} \frac{1}{M^2} \sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \frac{-i\omega m_{\chi,\mathbf{k}}^{(j)}}{\zeta_j^2 (\sin^2 k + \sin^2 k_0) + (m_{\chi,\mathbf{k}}^{(j)})^2} ,
\end{aligned} \tag{4.9}$$

with $m_{\chi,\mathbf{k}}^{(j)} = \sigma_\chi(\mathbf{k}) + (-1)^j \mu(\mathbf{k})$ and $\zeta_j = 1 + (-1)^j (\mu/2)$. Calling $m_\chi^{(j)} \stackrel{def}{=} m_{\chi,\mathbf{0}}^{(j)}$ one can easily verify that $m_\chi^{(j)}$ is given by (4.2) and the propagators are bounded as in (4.1), for some $\kappa > 0$. The similar equations and bounds for the ψ propagators are proven in the same way.

Calling

$$-\bar{\mathcal{V}}(\psi, \chi) = Q(\psi, \chi) - \nu F_\sigma(\psi) + \tilde{\lambda} V(\psi, \chi) , \tag{4.10}$$

and using the rules in Appendix A1, we obtain

$$M^2 \tilde{E}_1 + Q^{(1)}(\psi) + \mathcal{V}^{(1)}(\psi) = -\log \int P(d\chi) e^{-\bar{\mathcal{V}}(\psi, \chi)} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \mathcal{E}_\chi^T(\bar{\mathcal{V}}; n) . \tag{4.11}$$

We label each one of the monomials in $\bar{\mathcal{V}}$ by an index v_i , so that each monomial in $\bar{\mathcal{V}}$ can be written as

$$\sum_{\mathbf{x}_{v_i}} K_{v_i}(\mathbf{x}_{v_i}) \prod_{f \in \tilde{P}_{v_i}} \psi_{\omega(f), \mathbf{x}(f)}^{\alpha(f)} \prod_{f \in P_{v_i}} \chi_{\omega(f), \mathbf{x}(f)}^{\alpha(f)} , \tag{4.12}$$

where \mathbf{x}_{v_i} is the total set of coordinates associated to v_i , $K_{v_i}(\mathbf{x}_{v_i})$ is a bounded compact support function and P_{v_i} and \tilde{P}_{v_i} are the set of indices labelling the χ or ψ -fields in the monomial v_i ; the labels $\alpha(f), \omega(f), \mathbf{x}(f)$ assume values in the sets $\{\pm\}$, $\{\pm 1\}$ and Λ_M respectively. We can write

$$\begin{aligned}
\mathcal{V}^{(1)}(\psi) &= \sum_{\tilde{P}_{v_0} \neq \emptyset} \mathcal{V}^{(1)}(\tilde{P}_{v_0}) \quad , \quad \mathcal{V}^{(1)}(\tilde{P}_{v_0}) = \sum_{\mathbf{x}_{v_0}} \left[\prod_{f \in \tilde{P}_{v_0}} \psi_{\omega(f), \mathbf{x}(f)}^{\alpha(f)} \right] K_{\tilde{P}_{v_0}}(\mathbf{x}_{v_0}) \\
K_{\tilde{P}_{v_0}}(\mathbf{x}_{v_0}) &= \sum_{s=1}^{\infty} \frac{1}{s!} \sum_{i_1, \dots, i_s}^* \mathcal{E}_\chi^T(\tilde{\chi}(P_{v_1}), \dots, \tilde{\chi}(P_{v_s})) \prod_{i=1}^s K_{v_i}(\mathbf{x}_{v_i}) ,
\end{aligned} \tag{4.13}$$

where $\tilde{\chi}(P_{v_i}) = \prod_{f \in P_{v_i}} \chi_{\mathbf{x}(f), \omega(f)}^{\alpha(f)}$ and the $*$ on the sum means that we are excluding the case v_1, \dots, v_s all come from $Q(\psi, \chi)$ (such terms will contribute, by definition, to $Q^{(1)}(\psi)$). Furthermore $\sum_{i_1, \dots, i_s} \leq c^s$, for some constant c , $\tilde{P}_{v_0} = \bigcup_i \tilde{P}_{v_i}$ and $\mathbf{x}_{v_0} = \bigcup_i \mathbf{x}_{v_i}$.

We use now a generalization of a well known expression for \mathcal{E}_χ^T [Le], proven in Appendix A2:

$$\mathcal{E}_\chi^T(\tilde{\chi}(P_{v_1}), \dots, \tilde{\chi}(P_{v_n})) = \sum_T \alpha_T \prod_{\ell \in T} g_\chi(f_\ell^1, f_\ell^2) \int dP_T(\mathbf{t}) \text{Pf } G^T(\mathbf{t}) \tag{4.14}$$

where:

- a) T is a set of lines forming an *anchored tree* between the cluster of points P_{v_1}, \dots, P_{v_s} i.e. T is a set of lines which becomes a tree if one identifies all the points in the same clusters;
- b) α_T is a sign (irrelevant for the subsequent bounds);
- c) given $\ell \in T$, let f_ℓ^1, f_ℓ^2 the field labels associated to the points connected by ℓ ; $g_\chi(f_\ell^1, f_\ell^2)$ is defined as:

$$g_\chi(f_\ell^1, f_\ell^2) \stackrel{def}{=} g_{\underline{a}(f_\ell^1), \underline{a}(f_\ell^2)}^\chi(\mathbf{x}(f_\ell^1) - \mathbf{x}(f_\ell^2)) \quad , \quad \underline{a}(f) = (\alpha(f), \omega(f)) ; \quad (4.15)$$

- d) $\mathbf{t} = \{t_{i,i'} \in [0, 1], 1 \leq i, i' \leq s\}$, $dP_T(\mathbf{t})$ is a probability measure with support on a set of \mathbf{t} such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i \in \mathbb{R}^n$ of unit norm;
- e) if $2n = \sum_{i=1}^s |P_{v_i}|$, then $G^T(\mathbf{t})$ is a $(2n - 2s + 2) \times (2n - 2s + 2)$ antisymmetrix matrix, whose elements are given by $G_{f,f'}^T = t_{i(f), i(f')} g_\chi(f, f')$, where: $f, f' \notin F_T$ and $F_T \stackrel{def}{=} \cup_{\ell \in T} \{f_\ell^1, f_\ell^2\}$; $i(f)$ is s.t. $f \in P_{i(f)}$;
- f) $\text{Pf } G^T$ is the *Pfaffian* of G^T ; given an antisymmetrix matrix $A_{ij} = -A_{ji}$, $i, j = 1, \dots, 2k$, its Pfaffian is defined as

$$\begin{aligned} \text{Pf } A &= \frac{1}{2^k k!} \sum_{\pi} (-1)^\pi A_{\pi(1)\pi(2)} \cdots A_{\pi(2k-1)\pi(2k)} \\ &= \int d\psi_1 \cdots d\psi_{2k} e^{-\frac{1}{2} \sum_{i,j} \psi_i A_{ij} \psi_j} , \end{aligned} \quad (4.16)$$

where in the first line π is a permutation of $\{1, \dots, 2k\}$ and $(-1)^\pi$ is its parity while, in the second line, ψ_1, \dots, ψ_{2k} are Grassmanian variables. A well known property is that $(\text{Pf } A)^2 = \det A$.

If $s = 1$ the sum over T is empty, but we can still use the above equation by interpreting the r.h.s. as 1 if P_{v_1} is empty, and $\text{Pf } G^T(P_{v_1})$ otherwise.

In order to bound $\text{Pf } G^T$ we first use $|\text{Pf } G^T| = \sqrt{|\det G^T|}$ and then, in order to bound the determinant, the *Gram-Hadamard inequality*, proven in Appendix A3, stating that, if M is a square matrix with elements M_{ij} of the form $M_{ij} = \langle A_i, B_j \rangle$, where A_i, B_j are vectors in a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$, then

$$|\det M| \leq \prod_i \|A_i\| \cdot \|B_i\| . \quad (4.17)$$

where $\|\cdot\|$ is the norm induced by the scalar product.

Let $\mathcal{H} = \mathbb{R}^n \otimes \mathcal{H}_0$, where \mathcal{H}_0 is the Hilbert space of complex four dimensional vectors $F(\mathbf{k}) = (F_1(\mathbf{k}), \dots, \dots, F_4(\mathbf{k}))$, $F_i(\mathbf{k})$ being a function on the set \mathcal{D}_M , with scalar product

$$\langle F, G \rangle = \sum_{i=1}^4 \frac{1}{M^2} \sum_{\mathbf{k}} F_i^*(\mathbf{k}) G_i(\mathbf{k}) . \quad (4.18)$$

It is easy to verify that

$$G_{f,f'} = t_{i(f), i(f')} g_\chi(f, f') = \langle \mathbf{u}_{i(f)} \otimes A_f, \mathbf{u}_{i(f')} \otimes B_{f'} \rangle , \quad (4.19)$$

where $\mathbf{u}_i \in \mathbb{R}^n$, $i = 1, \dots, n$, are vectors such that $t_{i,i'} = \mathbf{u}_i \cdot \mathbf{u}_{i'}$, and, if $\hat{g}_{\underline{a}, \underline{a}'}^\chi(\mathbf{k})$ is the Fourier transform of $g_{\underline{a}, \underline{a}'}^\chi(\mathbf{x} - \mathbf{y})$, $A_f(\mathbf{k})$ and $B_{f'}(\mathbf{k})$ are given by

$$\begin{aligned} A_f(\mathbf{k}) &= e^{-i\mathbf{k}\mathbf{x}(f)} \left(\hat{g}_{\underline{a}(f), (-,1)}^\chi(\mathbf{k}), \hat{g}_{\underline{a}(f), (-,-1)}^\chi(\mathbf{k}), \hat{g}_{\underline{a}(f), (+,1)}^\chi(\mathbf{k}), \hat{g}_{\underline{a}(f), (+,-1)}^\chi(\mathbf{k}) \right) , \\ B_{f'}(\mathbf{k}) &= e^{-i\mathbf{k}\mathbf{x}(f')} \begin{cases} (1, 0, 0, 0), & \text{if } \underline{a}(f') = (-, 1), \\ (0, 1, 0, 0), & \text{if } \underline{a}(f') = (-, -1), \\ (0, 0, 1, 0), & \text{if } \underline{a}(f') = (+, 1), \\ (0, 0, 0, 1), & \text{if } \underline{a}(f') = (+, -1), \end{cases} \end{aligned} \quad (4.20)$$

Note that $\|A_f\| \leq C$, for some $C = O(1)$, and $\|B_{f'}\| = 4$. Hence we have proved that

$$|\text{Pf } G^T| = \sqrt{|\det G^T|} \leq c^s, \quad (4.21)$$

for some $c = O(1)$ (we used that $2n \leq 4s$). Finally we get

$$\sum_{\mathbf{x}_{v_0}} |K_{\tilde{P}_{v_0}}(\mathbf{x}_{v_0})| \leq \sum_{s=1}^{\infty} \frac{c^s}{s!} \sum_{v_1, \dots, v_s} \sum_{\mathbf{x}_{v_1}, \dots, \mathbf{x}_{v_s}} \sum_T \prod_{\ell \in T} |g_\chi(f_\ell^1, f_\ell^2)| \prod_{i=1}^s |K_i(\mathbf{x}_{v_i})| \quad (4.22)$$

where we have used that $\int dP_T(\mathbf{t}) = 1$. The number of addenda in \sum_T is bounded by $s!c^s$. Finally T and the $\bigcup_i \mathbf{x}_{v_i}$ form a tree connecting all points, so that, using that the propagators decay exponentially on scale $O(1)$ and that the interactions are short ranged, we find that, if $|\nu| \leq c|\lambda|$,

$$\sum_{v_1, \dots, v_s} \sum_{\mathbf{x}_{v_1}, \dots, \mathbf{x}_{v_s}} \sum_T \prod_{\ell \in T} |g_\chi(f_\ell^1, f_\ell^2)| \prod_{i=1}^s |K_i(\mathbf{x}_{v_i})| \leq c^s s! |\lambda|^m M^2, \quad (4.23)$$

where m is the number of couplings $O(\lambda, \nu)$ ($m \geq 1$ by construction).

Note that if v_i only come from $-\overline{V}(\psi, \chi) - Q(\psi, \chi)$, then $m = s$. Let us consider now the case in which there are n_0 end-points associated to $Q(\psi, \chi)$, which have $O(1)$ coupling. In this case $n_0 \leq |\tilde{P}_{v_0}|$. In fact in $Q(\psi, \chi)$ there are only terms of the form $\psi_{\mathbf{x}} \chi_{\mathbf{x}'}$, where \mathbf{x}' is either \mathbf{x} or $\mathbf{x} \pm \hat{e}_0$ or $\mathbf{x} \pm \hat{e}_1$, so at most the number of them is equal to the number of ψ fields. If we call $n_\lambda \leq m$ the number of vertices quartic in the fields it is clear that $n_\lambda \geq \max\{1, |\tilde{P}_{v_0}|/2 - 1\}$. Hence

$$\sum_{\mathbf{x}_{v_0}} |K_{\tilde{P}_{v_0}}(\mathbf{x}_{v_0})| \leq M^2 \sum_{n_0=0}^{|\tilde{P}_{v_0}|} c^{n_0} \sum_{m=1}^{\infty} c^m |\lambda|^{m/2} |\lambda|^{\max\{1/2, |\tilde{P}_{v_0}|/4 - 1/2\}} \quad (4.24)$$

The last bound implies that the kernels $\widehat{W}_{2n, \underline{\alpha}, \underline{\omega}}^{(1)}$ in (4.5), which are the Fourier transforms of $K_{\tilde{P}_{v_0}}(\mathbf{x}_{v_0})$, see (4.13), can be bounded as:

$$|\widehat{W}_{2n, \underline{\alpha}, \underline{\omega}}^{(1)}(\mathbf{k}_1, \dots, \mathbf{k}_{2n-1})| \leq M^2 C^n |\lambda|^{\max\{1, n/2\}}; \quad (4.25)$$

We now turn to the construction of $P_{Z_1, \sigma_1, \mu_1, C_1}$. We define:

$$e^{-t_1 M^2} P_{Z_1, \sigma_1, \mu_1, C_1}(d\psi) \stackrel{\text{def}}{=} P_\sigma(d\psi) e^{-Q^{(1)}(\psi)}, \quad (4.26)$$

where t_1 is chosen in such a way $\int P_{Z_1, \sigma_1, \mu_1, C_1}(d\psi) = 1$. From definition (4.26), (4.3) follows, with $E_1 = \tilde{E}_1 + t_1$ (\tilde{E}_1 was defined in (4.6)) and $\mathcal{V}^{(1)}(\psi)$ constructed above.

Let us now study in more detail the structure of $P_{Z_1, \sigma_1, \mu_1, C_1}(d\psi)$. In order to write it as an exponential of a quadratic form, it is sufficient to calculate the correlations

$$\begin{aligned} \langle \psi_{\omega_1, \mathbf{k}}^{\alpha_1} \psi_{\omega_2, -\alpha_1 \alpha_2 \mathbf{k}}^{\alpha_2} \rangle_1 &\stackrel{\text{def}}{=} \int P_{Z_1, \sigma_1, \mu_1, C_1}(d\psi) \psi_{\omega_1, \mathbf{k}}^{\alpha_1} \psi_{\omega_2, -\alpha_1 \alpha_2 \mathbf{k}}^{\alpha_2} = \\ &= e^{-t_1 M^2} \int P_\sigma(d\psi) P(d\chi) e^{Q(\chi, \psi)} \psi_{\omega_1, \mathbf{k}}^{\alpha_1} \psi_{\omega_2, -\alpha_1 \alpha_2 \mathbf{k}}^{\alpha_2}. \end{aligned} \quad (4.27)$$

It is easy to realize that the measure $\sim P_\sigma(d\psi) P(d\chi) e^{Q(\chi, \psi)}$ factorizes into the product of two measures generated by the fields $\psi_{\omega, \mathbf{x}}^{(j)}$, $j = 1, 2$, defined by $\psi_{\omega, \mathbf{x}}^\alpha = (\psi_{\omega, \mathbf{x}}^{(1)} + i(-1)^\alpha \psi_{\omega, \mathbf{x}}^{(2)})/\sqrt{2}$. In fact, using this change of variables, one finds that

$$P_\sigma(d\psi) P(d\chi) e^{Q(\chi, \psi)} = \prod_{j=1,2} \overline{P}^{(j)}(d\psi^{(j)}, d\chi^{(j)}), \quad (4.28)$$

with

$$\overline{P}^{(j)}(d\psi^{(j)}, d\chi^{(j)}) \stackrel{def}{=} \frac{1}{\mathcal{N}^{(j)}} \prod_{\mathbf{x}} d\psi_{\mathbf{x}}^{(j)} d\overline{\psi}_{\mathbf{x}}^{(j)} d\chi_{\mathbf{x}}^{(j)} d\overline{\chi}_{\mathbf{x}}^{(j)} e^{\sum_{\mathbf{x}} [S_{\nu, \mathbf{x}}^{(j, \psi)} + S_{\mathbf{x}}^{(j, \chi)} + Q_{\mathbf{x}}^{(j)}]}, \quad (4.29)$$

with $j = 1, 2$, $S_{\mathbf{x}}^{(j, \chi)}$, $Q_{\mathbf{x}}^{(j)}$ defined as in (3.25), (3.26) and

$$S_{\nu, \mathbf{x}}^{(j, \psi)} = S_{\mathbf{x}}^{(j, \psi)} + i(t_{\psi} - \sqrt{2} + 1) \overline{\psi}_{\mathbf{x}}^{(j)} \psi_{\mathbf{x}}^{(j)}, \quad (4.30)$$

where $S_{\mathbf{x}}^{(j, \psi)}$ is defined as in (3.23). Substituting these expressions in (4.29), we find that, if $\xi_{\mathbf{k}}^{(j), T} \stackrel{def}{=} (\psi_{\mathbf{k}}^{(j)}, \overline{\psi}_{\mathbf{k}}^{(j)}, \chi_{\mathbf{k}}^{(j)}, \overline{\chi}_{\mathbf{k}}^{(j)})$,

$$\begin{aligned} \overline{P}^{(j)}(d\psi^{(j)}, d\chi^{(j)}) &= \frac{1}{\mathcal{N}^{(j)}} \exp\left\{-\frac{t_{\lambda}^{(j)}}{4M^2} \sum_{\mathbf{k}} \xi_{\mathbf{k}}^{(j), T} C_{\mathbf{k}}^{(j)} \xi_{-\mathbf{k}}^{(j)}\right\} \\ C_{\mathbf{k}}^{(j)} &\stackrel{def}{=} \begin{pmatrix} -i \sin k - \sin k_0 & -im_{\psi, \mathbf{k}}^{(j)} & i \sin k - \sin k_0 & i(\cos k - \cos k_0) \\ im_{\psi, \mathbf{k}}^{(j)} & -i \sin k + \sin k_0 & -i(\cos k - \cos k_0) & i \sin k + \sin k_0 \\ i \sin k - \sin k_0 & i(\cos k - \cos k_0) & -i \sin k - \sin k_0 & -im_{\chi, \mathbf{k}}^{(j)} \\ -i(\cos k - \cos k_0) & i \sin k + \sin k_0 & im_{\chi, \mathbf{k}}^{(j)} & -i \sin k + \sin k_0 \end{pmatrix}. \end{aligned} \quad (4.31)$$

A lengthy but straightforward calculation shows that the determinant $B^{(j)}(\mathbf{k}) \stackrel{def}{=} \det C_{\mathbf{k}}^{(j)}$ is equal to

$$B^{(j)}(\mathbf{k}) = \frac{16}{(t_{\lambda}^{(j)})^4} \{2t_{\lambda}^{(j)} [1 - (t_{\lambda}^{(j)})^2] (2 - \cos k - \cos k_0) + (t_{\lambda}^{(j)} - t_{\psi})^2 (t_{\lambda}^{(j)} - t_{\chi})^2\} \quad (4.32)$$

Using, for $l, m = 1, \dots, 4$, the algebraic identity

$$\frac{1}{\mathcal{N}^{(j)}} \int \left[\prod_{\mathbf{k}, i} (d\xi_{\mathbf{k}}^{(j)})_i \right] (\xi_{-\mathbf{k}'}^{(j)})_l (\xi_{\mathbf{k}'}^{(j)})_m \exp\left\{-\frac{t_{\lambda}^{(j)}}{4M^2} \sum_{\mathbf{k}} \xi_{\mathbf{k}}^{(j), T} C_{\mathbf{k}}^{(j)} \xi_{-\mathbf{k}}^{(j)}\right\} = \frac{4M^2}{t_{\lambda}^{(j)}} (C_{\mathbf{k}'}^{(j)})_{lm}^{-1}, \quad (4.33)$$

we find:

$$\begin{aligned} \langle \psi_{-\mathbf{k}}^{(j)} \psi_{\mathbf{k}}^{(j)} \rangle_1 &= \frac{4M^2}{t_{\lambda}^{(j)}} \frac{c_{1,1}^{(j)}(\mathbf{k})}{B^{(j)}(\mathbf{k})}, \quad \langle \overline{\psi}_{-\mathbf{k}}^{(j)} \psi_{\mathbf{k}}^{(j)} \rangle_1 = \frac{4M^2}{t_{\lambda}^{(j)}} \frac{c_{-1,1}^{(j)}(\mathbf{k})}{B^{(j)}(\mathbf{k})}, \\ \langle \overline{\psi}_{-\mathbf{k}}^{(j)} \overline{\psi}_{\mathbf{k}}^{(j)} \rangle_1 &= \frac{4M^2}{t_{\lambda}^{(j)}} \frac{c_{-1,-1}^{(j)}(\mathbf{k})}{B^{(j)}(\mathbf{k})}, \end{aligned} \quad (4.34)$$

where, if $\omega = \pm 1$, recalling that $t_{\psi} = \sqrt{2} - 1 + \nu/2$ and $t_{\chi} = -\sqrt{2} - 1$,

$$\begin{aligned} c_{\omega, \omega}^{(j)}(\mathbf{k}) &\stackrel{def}{=} \frac{4}{(t_{\lambda}^{(j)})^2} \{2t_{\lambda}^{(j)} t_{\chi} (-i \sin k \cos k_0 + \omega \sin k_0 \cos k) + [(t_{\lambda}^{(j)})^2 + t_{\chi}^2] (i \sin k - \omega \sin k_0)\} \\ c_{\omega, -\omega}^{(j)}(\mathbf{k}) &\stackrel{def}{=} -i\omega \frac{4}{(t_{\lambda}^{(j)})^2} \left\{ -t_{\lambda}^{(j)} (3t_{\chi} + t_{\psi}) \cos k \cos k_0 + [(t_{\lambda}^{(j)})^2 + 2t_{\chi} t_{\psi} + t_{\chi}^2] (\cos k + \cos k_0) - \right. \\ &\quad \left. - (t_{\lambda}^{(j)} (t_{\psi} + t_{\chi}) + 2 \frac{t_{\psi} t_{\chi}^2}{t_{\lambda}^{(j)}}) \right\}. \end{aligned} \quad (4.35)$$

$P_{Z_1, \sigma_1, \mu_1, C_1}(d\psi)$ can now be written in terms of these correlations, as

$$P_{Z_1, \sigma_1, \mu_1, C_1}(d\psi) = P^{(1)}(d\psi^{(1)}) P^{(2)}(d\psi^{(2)}), \quad (4.36)$$

with

$$P^{(j)}(d\psi^{(j)}) \stackrel{def}{=} \frac{1}{N_j} \prod_{\mathbf{k}} d\psi_{\mathbf{k}}^{(j)} d\bar{\psi}_{\mathbf{k}}^{(j)} \cdot \exp\left\{-\frac{t_{\lambda}^{(j)} B^{(j)}(\mathbf{k})}{4M^2 \det c_{\mathbf{k}}^{(j)}} (\psi_{\mathbf{k}}^{(j)}, \bar{\psi}_{\mathbf{k}}^{(j)}) \begin{pmatrix} c_{-1,-1}^{(j)}(\mathbf{k}) & -c_{1,-1}^{(j)}(\mathbf{k}) \\ -c_{-1,1}^{(j)}(\mathbf{k}) & c_{1,1}^{(j)}(\mathbf{k}) \end{pmatrix} \begin{pmatrix} \psi_{-\mathbf{k}}^{(j)} \\ \bar{\psi}_{-\mathbf{k}}^{(j)} \end{pmatrix}\right\}, \quad (4.37)$$

where $\det c_{\mathbf{k}}^{(j)} = c_{1,1}^{(j)}(\mathbf{k})c_{-1,-1}^{(j)}(\mathbf{k}) - c_{1,-1}^{(j)}(\mathbf{k})c_{-1,1}^{(j)}(\mathbf{k})$. If we now use the identity $t_{\lambda}^{(j)} = t_{\psi}(2 + (-1)^j \mu)/(2 - \sigma)$ and rewrite the measure $P^{(1)}(d\psi^{(1)})P^{(2)}(d\psi^{(2)})$ in terms of $\psi_{\omega,\mathbf{k}}^{\pm}$ we find:

$$P_{Z_1, \sigma_1, \mu_1, C_1}(d\psi) = \frac{1}{N^{(1)}} \prod_{\mathbf{k}, \omega} d\psi_{\omega,\mathbf{k}}^{+} d\psi_{\omega,\mathbf{k}}^{-} \exp\left\{-\frac{Z_1 C_1(\mathbf{k})}{4M^2} \Psi_{\mathbf{k}}^{+,T} A_{\psi}^{(1)} \Psi_{\mathbf{k}}^{-}\right\}, \quad (4.38)$$

with $C_1(\mathbf{k})$, Z_1 , σ_1 and μ_1 defined as after (4.3), and $A_{\psi}^{(1)}(\mathbf{k})$ as in (4.4), with

$$M^{(1)}(\mathbf{k}) = \frac{2}{2 - \sigma} \begin{pmatrix} -c_{-1,-1}^{+}(\mathbf{k}) & c_{-1,1}^{+}(\mathbf{k}) \\ c_{1,-1}^{+}(\mathbf{k}) & -c_{1,1}^{+}(\mathbf{k}) \end{pmatrix}, \quad N^{(1)}(\mathbf{k}) = \frac{2}{2 - \sigma} \begin{pmatrix} -c_{-1,-1}^{-}(\mathbf{k}) & c_{-1,1}^{-}(\mathbf{k}) \\ c_{1,-1}^{-}(\mathbf{k}) & -c_{1,1}^{-}(\mathbf{k}) \end{pmatrix}, \quad (4.39)$$

where $c_{\omega_1, \omega_2}^{\alpha}(\mathbf{k}) \stackrel{def}{=} [(1 - \mu/2)B^{(1)}(\mathbf{k})c_{\omega_1, \omega_2}^{(1)}(\mathbf{k})/\det c_{\mathbf{k}}^{(1)} + \alpha(1 + \mu/2)B^{(2)}(\mathbf{k})c_{\omega_1, \omega_2}^{(2)}(\mathbf{k})/\det c_{\mathbf{k}}^{(2)}]/2$. It is easy to verify that $A_{\psi}^{(1)}(\mathbf{k})$ can be written in the same form as (4.4). In fact, computing the functions in (4.39), one finds that, for \mathbf{k} , σ_1 and μ_1 small,

$$\begin{aligned} M^{(1)}(\mathbf{k}) &= \begin{pmatrix} (i \sin k + \sin k_0)(1 + O(\sigma_1)) + O(\mathbf{k}^3) & -i\sigma_1 + O(\mathbf{k}^2) \\ i\sigma_1 + O(\mathbf{k}^2) & (i \sin k - \sin k_0)(1 + O(\sigma_1)) + O(\mathbf{k}^3) \end{pmatrix} \\ N^{(1)}(\mathbf{k}) &= \begin{pmatrix} (i \sin k + \sin k_0)O(\mu_1) + O(\mathbf{k}^3) & i\mu_1 + O(\mu_1 \mathbf{k}^2) \\ -i\mu_1 + O(\mu_1 \mathbf{k}^2) & (i \sin k - \sin k_0)O(\mu_1) + O(\mathbf{k}^3) \end{pmatrix}, \end{aligned} \quad (4.40)$$

where the higher order terms in \mathbf{k} , σ_1 and μ_1 contribute to the corrections $a_1^{\pm}(\mathbf{k})$, $b_1^{\pm}(\mathbf{k})$, $c_1(\mathbf{k})$ and $d_1(\mathbf{k})$.

4.3. Symmetry properties.

In this section we identify some symmetries of model (3.19) and, using these symmetry properties, we prove that the quadratic and quartic terms in $\mathcal{V}^{(1)}$ and the corrections $a_1^{\pm}(\mathbf{k})$, $b_1^{\pm}(\mathbf{k})$, $c_1(\mathbf{k})$ and $d_1(\mathbf{k})$ appearing in (4.4) have a special structure, described in Theorem 4.1 below.

We start with noting that the formal action appearing in (3.19) (see also (2.14), (2.36) and (3.20) for an explicit form of the different contributions appearing in (3.19)) is invariant under the following transformations.

1) *Parity*:

$$H_{\mathbf{x}}^{(j)} \rightarrow \bar{H}_{-\mathbf{x}}^{(j)}, \quad \bar{H}_{\mathbf{x}}^{(j)} \rightarrow -H_{-\mathbf{x}}^{(j)}, \quad V_{\mathbf{x}}^{(j)} \rightarrow \bar{V}_{-\mathbf{x}}^{(j)}, \quad \bar{V}_{\mathbf{x}}^{(j)} \rightarrow -V_{-\mathbf{x}}^{(j)}. \quad (4.41)$$

In terms of the variables $\hat{\psi}_{\omega,\mathbf{k}}^{\alpha}$, this transformation is equivalent to $\hat{\psi}_{\omega,\mathbf{k}}^{\alpha} \rightarrow i\omega \hat{\psi}_{\omega,-\mathbf{k}}^{\alpha}$ (the same for χ) and we shall call it *parity*.

2) *Complex conjugation*:

$$\psi_{\mathbf{x}}^{(j)} \rightarrow \bar{\psi}_{\mathbf{x}}^{(j)}, \quad \bar{\psi}_{\mathbf{x}}^{(j)} \rightarrow \psi_{\mathbf{x}}^{(j)}, \quad \chi_{\mathbf{x}}^{(j)} \rightarrow \bar{\chi}_{\mathbf{x}}^{(j)}, \quad \bar{\chi}_{\mathbf{x}}^{(j)} \rightarrow \chi_{\mathbf{x}}^{(j)}, \quad c \rightarrow c^*, \quad (4.42)$$

where c is a generic constant appearing in the formal action and c^* is its complex conjugate. In terms of the variables $\hat{\psi}_{\omega,\mathbf{k}}^{\alpha}$, this transformation is equivalent to $\hat{\psi}_{\omega,\mathbf{k}}^{\alpha} \rightarrow \hat{\psi}_{-\omega,\mathbf{k}}^{-\alpha}$ (the same for χ), $c \rightarrow c^*$ and we shall call

it *complex conjugation*.

3) *Hole-particle*:

$$\begin{aligned} H_{\mathbf{x}}^{(j)} &\rightarrow (-1)^{j+1} H_{\mathbf{x}}^{(j)} , & \overline{H}_{\mathbf{x}}^{(j)} &\rightarrow (-1)^{j+1} \overline{H}_{\mathbf{x}}^{(j)} , \\ V_{\mathbf{x}}^{(j)} &\rightarrow (-1)^{j+1} V_{\mathbf{x}}^{(j)} , & \overline{V}_{\mathbf{x}}^{(j)} &\rightarrow (-1)^{j+1} \overline{V}_{\mathbf{x}}^{(j)} . \end{aligned} \quad (4.43)$$

This transformation is equivalent to $\hat{\psi}_{\omega, \mathbf{k}}^{\alpha} \rightarrow \hat{\psi}_{\omega, -\mathbf{k}}^{-\alpha}$ (the same for χ) and we shall call it *hole-particle*.

4) *Rotation*:

$$\begin{aligned} H_{x, x_0}^{(j)} &\rightarrow i \overline{V}_{-x_0, -x}^{(j)} , & \overline{H}_{x, x_0}^{(j)} &\rightarrow i V_{-x_0, -x}^{(j)} , \\ V_{x, x_0}^{(j)} &\rightarrow i \overline{H}_{-x_0, -x}^{(j)} , & \overline{V}_{x, x_0}^{(j)} &\rightarrow i H_{-x_0, -x}^{(j)} . \end{aligned} \quad (4.44)$$

This transformation is equivalent to

$$\hat{\psi}_{\omega, (k, k_0)}^{\alpha} \rightarrow -\omega e^{-i\omega\pi/4} \hat{\psi}_{-\omega, (-k_0, -k)}^{\alpha} , \quad \hat{\chi}_{\omega, (k, k_0)}^{\alpha} \rightarrow \omega e^{-i\omega\pi/4} \hat{\chi}_{-\omega, (-k_0, -k)}^{\alpha} \quad (4.45)$$

and we shall call it *rotation*.

5) *Reflection*:

$$\begin{aligned} H_{x, x_0}^{(j)} &\rightarrow i \overline{H}_{-x, x_0}^{(j)} , & \overline{H}_{x, x_0}^{(j)} &\rightarrow i H_{-x, x_0}^{(j)} , \\ V_{x, x_0}^{(j)} &\rightarrow -i V_{-x, x_0}^{(j)} , & \overline{V}_{x, x_0}^{(j)} &\rightarrow i \overline{V}_{-x, x_0}^{(j)} . \end{aligned} \quad (4.46)$$

This transformation is equivalent to $\hat{\psi}_{\omega, (k, k_0)}^{\alpha} \rightarrow i \hat{\psi}_{-\omega, (-k, k_0)}^{\alpha}$ (the same for χ) and we shall call it *reflection*.

6) *The (1) \longleftrightarrow (2) symmetry*.

$$\begin{aligned} H_{\mathbf{x}}^{(1)} &\longleftrightarrow H_{\mathbf{x}}^{(2)} , & \overline{H}_{\mathbf{x}}^{(1)} &\longleftrightarrow \overline{H}_{\mathbf{x}}^{(2)} , \\ V_{\mathbf{x}}^{(1)} &\longleftrightarrow V_{\mathbf{x}}^{(2)} , & \overline{V}_{\mathbf{x}}^{(1)} &\longleftrightarrow \overline{V}_{\mathbf{x}}^{(2)} , & u &\rightarrow -u . \end{aligned} \quad (4.47)$$

This transformation is equivalent to $\hat{\psi}_{\omega, \mathbf{k}}^{\alpha} \rightarrow -i\alpha \hat{\psi}_{\omega, -\mathbf{k}}^{-\alpha}$ (the same for χ) together with $u \rightarrow -u$ and we shall call it (1) \longleftrightarrow (2) *symmetry*.

It is easy to verify that the quadratic forms $P(d\chi)$, $P(d\psi)$ and $P_{Z_1, \sigma_1, \mu_1, C_1}(d\psi)$ are separately invariant under the symmetries above. Then the effective action $\mathcal{V}^{(1)}(\psi)$ is still invariant under the same symmetries. Using the invariance of $\mathcal{V}^{(1)}$ under transformations (1)–(6), we now study in detail the structure of its quadratic and quartic terms.

Quartic term. Let us consider in (4.5) the term with $2n = 4$, $\alpha_1 = \alpha_2 = -\alpha_3 = -\alpha_4 = +$, $\omega_1 = -\omega_2 = \omega_3 = -\omega_4 = 1$; for simplicity of notation, let us denote it with $\sum_{\mathbf{k}_i} W(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \hat{\psi}_{1, \mathbf{k}_1}^+ \hat{\psi}_{-1, \mathbf{k}_2}^+ \hat{\psi}_{-1, \mathbf{k}_3}^- \hat{\psi}_{1, \mathbf{k}_4}^- \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)$. Under complex conjugation it becomes equal to $\sum_{\mathbf{k}_i} W^*(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \hat{\psi}_{-1, \mathbf{k}_1}^- \hat{\psi}_{1, \mathbf{k}_2}^- \hat{\psi}_{1, \mathbf{k}_3}^+ \hat{\psi}_{-1, \mathbf{k}_4}^+ \delta(\mathbf{k}_3 + \mathbf{k}_4 - \mathbf{k}_1 - \mathbf{k}_2)$, so that $W(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = W^*(\mathbf{k}_3, \mathbf{k}_4, \mathbf{k}_1, \mathbf{k}_2)$.

Then, defining $L_1 = W(\bar{\mathbf{k}}_{++}, \bar{\mathbf{k}}_{++}, \bar{\mathbf{k}}_{++}, \bar{\mathbf{k}}_{++})$, where $\bar{\mathbf{k}}_{++} = (\pi/M, \pi/M)$, and $l_1 = \mathcal{P}_0 L_1 \stackrel{def}{=} L_1|_{\sigma_1 = \mu_1 = 0}$, we see that L_1 and l_1 are real. From the explicit computation of the lower order term we find $l_1 = \tilde{\lambda}/Z_1^2 + O(\lambda^2)$.

Quadratic terms. We distinguish 4 cases (items (a)–(d) below).

a) Let us consider in (4.5) the term with $2n = 2$, $\alpha_1 = -\alpha_2 = +$ and $\omega_1 = -\omega_2 = \omega$; let us denote it with

$\sum_{\omega, \mathbf{k}} W_{\omega}(\mathbf{k}; \mu_1) \hat{\psi}_{\omega, \mathbf{k}}^+ \hat{\psi}_{-\omega, \mathbf{k}}^-$. Under parity it becomes

$$\sum_{\omega, \mathbf{k}} W_{\omega}(\mathbf{k}; \mu_1) (i\omega) \hat{\psi}_{\omega, -\mathbf{k}}^+ (-i\omega) \hat{\psi}_{-\omega, -\mathbf{k}}^- = \sum_{\omega, \mathbf{k}} W_{\omega}(-\mathbf{k}; \mu_1) \hat{\psi}_{\omega, \mathbf{k}}^+ \hat{\psi}_{-\omega, \mathbf{k}}^-, \quad (4.48)$$

so that $W_{\omega}(\mathbf{k}; \mu_1)$ is even in \mathbf{k} .

Under complex conjugation it becomes

$$\sum_{\omega, \mathbf{k}} W_{\omega}(\mathbf{k}; \mu_1)^* \hat{\psi}_{-\omega, \mathbf{k}}^- \hat{\psi}_{\omega, \mathbf{k}}^+ = - \sum_{\omega, \mathbf{k}} W_{\omega}(\mathbf{k}; \mu_1)^* \hat{\psi}_{\omega, \mathbf{k}}^+ \hat{\psi}_{-\omega, \mathbf{k}}^-, \quad (4.49)$$

so that $W_{\omega}(\mathbf{k}; \mu_1)$ is purely imaginary.

Under hole-particle it becomes

$$\sum_{\omega, \mathbf{k}} W_{\omega}(\mathbf{k}; \mu_1) \hat{\psi}_{\omega, -\mathbf{k}}^- \hat{\psi}_{-\omega, -\mathbf{k}}^+ = - \sum_{\omega, \mathbf{k}} W_{-\omega}(\mathbf{k}; \mu_1) \hat{\psi}_{\omega, \mathbf{k}}^+ \hat{\psi}_{-\omega, \mathbf{k}}^-, \quad (4.50)$$

so that $W_{\omega}(\mathbf{k}; \mu_1)$ is odd in ω .

Under $(1) \longleftrightarrow (2)$ it becomes:

$$\sum_{\omega, \mathbf{k}} W_{\omega}(\mathbf{k}; -\mu_1) (-i) \hat{\psi}_{-\omega, -\mathbf{k}}^- (i) \hat{\psi}_{\omega, -\mathbf{k}}^+ = \sum_{\omega, \mathbf{k}} W_{\omega}(\mathbf{k}; -\mu_1) \hat{\psi}_{\omega, \mathbf{k}}^+ \hat{\psi}_{-\omega, \mathbf{k}}^-, \quad (4.51)$$

so that $W_{\omega}(\mathbf{k}; \mu_1)$ is even in μ_1 . Let us define $S_1 = i\omega/2 \sum_{\eta, \eta'} W_{\omega}(\bar{\mathbf{k}}_{\eta\eta'})$, where $\bar{\mathbf{k}}_{\eta\eta'} = (\eta\pi/M, \eta'\pi/M)$, and $\gamma n_1 = \mathcal{P}_0 S_1$, $s_1 = \mathcal{P}_1 S_1 = \sigma_1 \partial_{\sigma_1} S_1|_{\sigma_1=\mu_1=0} + \mu_1 \partial_{\mu_1} S_1|_{\sigma_1=\mu_1=0}$. From the previous discussion we see that S_1, s_1 and n_1 are real and s_1 is independent of μ_1 . From the computation of the lower order terms we find $s_1 = O(\lambda\sigma_1)$ and $\gamma n_1 = \nu/Z_1 + c_1'\lambda + O(\lambda^2)$, for some constant c_1' independent of λ . Note that, since $W_{\omega}(\mathbf{k}; \mu_1)$ is even in \mathbf{k} (so that in particular no linear terms in \mathbf{k} appear) in real space no terms of the form $\psi_{\omega, \mathbf{x}}^+ \partial \psi_{-\omega, \mathbf{x}}^-$ can appear.

b) Let us consider in (4.5) the term with $2n = 2$, $\alpha_1 = \alpha_2 = \alpha$ and $\omega_1 = -\omega_2 = \omega$ and let us denote it with $\sum_{\omega, \alpha, \mathbf{k}} W_{\omega}^{\alpha}(\mathbf{k}; \mu_1) \hat{\psi}_{\omega, \mathbf{k}}^{\alpha} \hat{\psi}_{-\omega, -\mathbf{k}}^{\alpha}$. We proceed as in item (a) and, by using parity, we see that $W_{\omega}^{\alpha}(\mathbf{k}; \mu_1)$ is even in \mathbf{k} and odd in ω .

By using complex conjugation, we see that $W_{\omega}^{\alpha}(\mathbf{k}; \mu_1) = -W_{-\omega}^{\alpha}(\mathbf{k}; \mu_1)^*$.

By using hole-particle, we see that $W_{\omega}^{\alpha}(\mathbf{k}; \mu_1)$ is even in α and $W_{\omega}^{\alpha}(\mathbf{k}; \mu_1) = -W_{\omega}^{-\alpha}(\mathbf{k}; \mu_1)^*$ implies that $W_{\omega}^{\alpha}(\mathbf{k}; \mu_1)$ is purely imaginary.

By using $(1) \longleftrightarrow (2)$ we see that $W_{\omega}^{\alpha}(\mathbf{k}; \mu_1)$ is odd in μ_1 .

If we define $M_1 = -i\omega/2 \sum_{\eta, \eta'} W_{\omega}^{\alpha}(\bar{\mathbf{k}}_{\eta\eta'}; \mu_1)$ and $m_1 = \mathcal{P}_1 M_1$, from the previous properties follows that M_1 and m_1 are real, m_1 is independent of σ_1 and, from the computation of its lower order, $m_1 = O(\lambda\mu_1)$. Note that, since $W_{\omega}^{\alpha}(\mathbf{k}; \mu_1)$ is even in \mathbf{k} (so that in particular no linear terms in \mathbf{k} appear) in real space no terms of the form $\psi_{\omega, \mathbf{x}}^{\alpha} \partial \psi_{-\omega, \mathbf{x}}^{\alpha}$ can appear.

c) Let us consider in (4.5) the term with $2n = 2$, $\alpha_1 = -\alpha_2 = +$, $\omega_1 = \omega_2 = \omega$ and let us denote it with $\sum_{\omega, \mathbf{k}} W_{\omega}(\mathbf{k}; \mu_1) \hat{\psi}_{\omega, \mathbf{k}}^+ \hat{\psi}_{\omega, \mathbf{k}}^-$. By using parity we see that $W_{\omega}(\mathbf{k}; \mu_1)$ is odd in \mathbf{k} .

By using reflection we see that $W_{\omega}(k, k_0; \mu_1) = W_{-\omega}(k, -k_0; \mu_1)$.

By using complex conjugation we see that $W_{\omega}(k, k_0; \mu_1) = W_{\omega}^*(-k, k_0; \mu_1)$.

By using rotation we find $W_{\omega}(k, k_0; \mu_1) = -i\omega W_{\omega}(k_0, -k; \mu_1)$.

By using $(1) \longleftrightarrow (2)$ we see that $W_{\omega}(\mathbf{k}; -\mu_1)$ is even in μ_1 .

We now define

$$G_1(\mathbf{k}) = \frac{1}{4} \sum_{\eta, \eta'} W_\omega(\bar{\mathbf{k}}_{\eta\eta'}; \mu_1) \left(\eta \frac{\sin k}{\sin \pi/M} + \eta' \frac{\sin k_0}{\sin \pi/M} \right).$$

We can rewrite $G_1(\mathbf{k}) = a_\omega \sin k + b_\omega \sin k_0$, with

$$\begin{aligned} a_\omega &= \frac{1}{2 \sin \frac{\pi}{M}} \left[W_\omega\left(\frac{\pi}{M}, \frac{\pi}{M}; \mu_1\right) + W_\omega\left(\frac{\pi}{M}, -\frac{\pi}{M}; \mu_1\right) \right] \\ b_\omega &= \frac{1}{2 \sin \frac{\pi}{M}} \left[W_\omega\left(\frac{\pi}{M}, \frac{\pi}{M}; \mu_1\right) - W_\omega\left(\frac{\pi}{M}, -\frac{\pi}{M}; \mu_1\right) \right]. \end{aligned} \quad (4.52)$$

From the properties of $W_\omega(\mathbf{k}; \mu_1)$ discussed above, we get:

$$\begin{aligned} W_\omega\left(\frac{\pi}{M}, \frac{\pi}{M}; \mu_1\right) &= W_{-\omega}\left(\frac{\pi}{M}, -\frac{\pi}{M}; \mu_1\right) = -W_\omega^*\left(\frac{\pi}{M}, -\frac{\pi}{M}; \mu_1\right) = -i\omega W_\omega\left(\frac{\pi}{M}, -\frac{\pi}{M}; \mu_1\right) \\ W_\omega\left(\frac{\pi}{M}, -\frac{\pi}{M}; \mu_1\right) &= W_{-\omega}\left(\frac{\pi}{M}, \frac{\pi}{M}; \mu_1\right) = -W_\omega^*\left(\frac{\pi}{M}, \frac{\pi}{M}; \mu_1\right) = i\omega W_\omega\left(\frac{\pi}{M}, \frac{\pi}{M}; \mu_1\right) \end{aligned} \quad (4.53)$$

so that

$$\begin{aligned} a_\omega &= a_{-\omega} = -a_\omega^* = i\omega b_\omega \stackrel{def}{=} ia \\ b_\omega &= -b_{-\omega} = b_\omega^* = -i\omega a_\omega \stackrel{def}{=} \omega b = -i\omega ia \end{aligned} \quad (4.54)$$

with $a = b$ real and independent of ω . As a consequence, $G_1(\mathbf{k}) = G_1(i \sin k + \omega \sin k_0)$ for some real constant G_1 . If $z_1 \stackrel{def}{=} \mathcal{P}_0 G_1$ and we compute the lowest order contribution to z_1 , we find $z_1 = O(\lambda^2)$.

d) Let us consider in (4.5) the term with $2n = 2$, $\alpha_1 = \alpha_2 = \alpha$, $\omega_1 = \omega_2 = \omega$ and let us denote it with $\sum_{\alpha, \omega, \mathbf{k}} W_\omega^\alpha(\mathbf{k}; \mu_1) \hat{\psi}_{\omega, \mathbf{k}}^\alpha \hat{\psi}_{\omega, -\mathbf{k}}^\alpha$. Repeating the proof in item c) we see that $W_\omega^\alpha(\mathbf{k}; \mu_1)$ is odd in \mathbf{k} and in μ_1 and, if we define

$$F_1(\mathbf{k}) = \frac{1}{4} \sum_{\eta, \eta'} W_\omega^\alpha(\bar{\mathbf{k}}_{\eta\eta'}; \mu_1) \left(\eta \frac{\sin k}{\sin \pi/M} + \eta' \frac{\sin k_0}{\sin \pi/M} \right),$$

we can rewrite $F_1(\mathbf{k}) = F_1(i \sin k + \omega \sin k_0)$. Since $W_\omega^\alpha(\mathbf{k}; \mu_1)$ is odd in μ_1 , we find $F_1 = O(\lambda \mu_1)$.

This concludes the study of the properties of the kernels of $\mathcal{V}^{(1)}$ we shall need in the following. Repeating the proof above it can also be seen that the corrections $a_1^\pm(\mathbf{k})$, $b_1^\pm(\mathbf{k})$, appearing in (4.4), are analytic odd functions of \mathbf{k} , while $c_1(\mathbf{k})$ and $d_1(\mathbf{k})$ are real and even; the explicit computation of the lower order terms in the Taylor expansion in \mathbf{k} shows that, in a neighborhood of $\mathbf{k} = \mathbf{0}$, $a_1^\pm(\mathbf{k}) = O(\sigma_1 \mathbf{k}) + O(\mathbf{k}^3)$, $b_1^\pm(\mathbf{k}) = O(\mu_1 \mathbf{k}) + O(\mathbf{k}^3)$, $c_1(\mathbf{k}) = O(\mathbf{k}^2)$ and $d_1(\mathbf{k}) = O(\mu_1 \mathbf{k}^2)$.

The result of the previous discussion can be collected in the following Theorem.

THEOREM 4.1 *Assume that $|\sigma_1|, |\mu_1| \leq c_1$ for some constant $c_1 > 0$. There exist a constant ε such that, if $|\lambda|, |\nu| \leq \varepsilon$, then Ξ_{AT}^- can be written as in (4.3), (4.4), (4.5), where:*

- 1) E_1 is an $O(1)$ constant;
- 2) $a_1^\pm(\mathbf{k}), b_1^\pm(\mathbf{k})$ are analytic odd functions of \mathbf{k} and $c_1(\mathbf{k}), d_1(\mathbf{k})$ real analytic even functions of \mathbf{k} ; in a neighborhood of $\mathbf{k} = \mathbf{0}$, $a_1^\pm(\mathbf{k}) = O(\sigma_1 \mathbf{k}) + O(\mathbf{k}^3)$, $b_1^\pm(\mathbf{k}) = O(\mu_1 \mathbf{k}) + O(\mathbf{k}^3)$, $c_1(\mathbf{k}) = O(\mathbf{k}^2)$ and $d_1(\mathbf{k}) = O(\mu_1 \mathbf{k}^2)$;
- 3) the determinant $|\det A_\psi^{(1)}(\mathbf{k})|$ can be bounded above and below by two positive constants times $[(\sigma_1 - \mu_1)^2 + |c(\mathbf{k})|][(\sigma_1 + \mu_1)^2 + |c(\mathbf{k})|]$ and $c(\mathbf{k}) = \cos k_0 + \cos k - 2$;
- 4) $\widehat{W}_{2n, \underline{\alpha}, \underline{\omega}}^{(1)}$ are analytic functions of $\mathbf{k}_i, \lambda, \nu, \sigma_1, \mu_1$, $i = 1, \dots, 2n$ and, for some constant C ,

$$|\widehat{W}_{2n, \underline{\alpha}, \underline{\omega}}^{(1)}(\mathbf{k}_1, \dots, \mathbf{k}_{2n-1})| \leq M^2 C^n |\lambda|^{\max\{1, n/2\}}; \quad (4.55)$$

4-a) the terms in (4.5) with $n = 2$ can be written as

$$\begin{aligned}
& L_1 \sum_{\mathbf{k}_1, \dots, \mathbf{k}_4} \hat{\psi}_{1, \mathbf{k}_1}^+ \hat{\psi}_{-1, \mathbf{k}_2}^+ \hat{\psi}_{-1, \mathbf{k}_3}^- \hat{\psi}_{1, \mathbf{k}_4}^- \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) + \\
& + \sum_{\mathbf{k}_1, \dots, \mathbf{k}_4} \sum_{\underline{\alpha}, \underline{\omega}} \widetilde{W}_{4, \underline{\alpha}, \underline{\omega}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \hat{\psi}_{\omega_1, \mathbf{k}_1}^{\alpha_1} \hat{\psi}_{\omega_2, \mathbf{k}_2}^{\alpha_2} \hat{\psi}_{\omega_3, \mathbf{k}_3}^{\alpha_3} \hat{\psi}_{\omega_4, \mathbf{k}_4}^{\alpha_4} \delta\left(\sum_{i=1}^4 \alpha_i \mathbf{k}_i\right),
\end{aligned} \tag{4.56}$$

where L_1 is real and $\widetilde{W}_{4, \underline{\alpha}, \underline{\omega}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ vanishes at $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}_3 = \left(\frac{\pi}{M}, \frac{\pi}{M}\right)$;

4-b) the term in (4.5) with $n = 1$ can be written as:

$$\begin{aligned}
& \frac{1}{4} \sum_{\omega, \alpha = \pm} \sum_{\mathbf{k}} \left[S_1(-i\omega) \hat{\psi}_{\omega, \mathbf{k}}^+ \hat{\psi}_{-\omega, \mathbf{k}}^- + M_1(i\omega) \hat{\psi}_{\omega, \mathbf{k}}^\alpha \hat{\psi}_{-\omega, -\mathbf{k}}^\alpha + F_1(i \sin k + \omega \sin k_0) \hat{\psi}_{\omega, \mathbf{k}}^\alpha \hat{\psi}_{\omega, -\mathbf{k}}^\alpha + \right. \\
& \left. + G_1(i \sin k + \omega \sin k_0) \hat{\psi}_{\omega, \mathbf{k}}^+ \hat{\psi}_{\omega, \mathbf{k}}^- \right] + \sum_{\mathbf{k}} \sum_{\underline{\alpha}, \underline{\omega}} \widetilde{W}_{2, \underline{\alpha}, \underline{\omega}}(\mathbf{k}) \hat{\psi}_{\omega_1, \mathbf{k}}^{\alpha_1} \hat{\psi}_{\omega_2, -\alpha_1 \alpha_2 \mathbf{k}}^{\alpha_2}
\end{aligned} \tag{4.57}$$

where: $\widetilde{W}_{2, \underline{\alpha}, \underline{\omega}}(\mathbf{k})$ is $O(\mathbf{k}^2)$ in a neighborhood of $\mathbf{k} = \mathbf{0}$; S_1, M_1, F_1, G_1 are real analytic functions of $\lambda, \sigma_1, \mu_1, \nu$ s.t. $F_1 = O(\lambda \mu_1)$ and

$$\begin{aligned}
L_1 &= l_1 + O(\lambda \sigma_1) + O(\lambda \mu_1) \quad , \quad S_1 = s_1 + \gamma n_1 + O(\lambda \sigma_1^2) + O(\lambda \mu_1^2) \\
M_1 &= m_1 + O(\lambda \mu_1 \sigma_1) + O(\lambda \mu_1^3) \quad , \quad G_1 = z_1 + O(\lambda \sigma_1) + O(\lambda \mu_1)
\end{aligned} \tag{4.58}$$

with $s_1 = \sigma_1 f_1$, $m_1 = \mu_1 f_2$ and l_1, n_1, f_1, f_2, z_1 independent of σ_1, μ_1 ; moreover $l_1 = \tilde{\lambda}/Z_1^2 + O(\lambda^2)$, $f_1, f_2 = O(\lambda)$, $\gamma n_1 = \nu/Z_1 + c_1' \lambda + O(\lambda^2)$, for some c_1' independent of λ , and $z_1 = O(\lambda^2)$.

Remark. The meaning of Theorem 2.1 is that after the integration of the χ fields we are left with a fermionic integration similar to (3.33) up to corrections which are at least $O(\mathbf{k}^2)$, and an effective interaction containing terms with any number of fields. *A priori* many bilinear terms with kernel $O(1)$ or $O(\mathbf{k})$ with respect to \mathbf{k} near $\mathbf{k} = \mathbf{0}$ could be generated by the χ -integration besides the ones originally present in (2.29); however *symmetry considerations restrict drastically the number of possible bilinear terms* $O(1)$ or $O(\mathbf{k})$. Only one new term of the form $\sum_{\mathbf{k}} (i \sin k + \omega \sin k_0) \hat{\psi}_{\omega, \mathbf{k}}^\alpha \hat{\psi}_{\omega, -\mathbf{k}}^\alpha$ appears, which is “dimensionally” *marginal* in a RG sense; however it is weighted by a constant $O(\lambda \mu_1)$ and this will improve its “dimension”, so that it will result to be *irrelevant*, see next Chapter.

5. Renormalization Group for light fermions. The anomalous regime.

In this Chapter we begin to describe the iterative integration scheme we shall follow in order to compute the Grassmann functional integral in (4.3). Each step of the iteration will resemble for many technical aspects the ultraviolet step described in the previous Chapter. We first split the light field ψ in a sum of independent Grassmann fields $\sum_h \psi_h$ with masses smaller and smaller, labeled by a *scale index* $h \leq 1$. Then we begin to integrate step by step each of them, starting from that with the biggest mass. After each integration step we rewrite the partition function in a way similar to the r.h.s. of (4.3), with new effective parameters Z_h, σ_h, μ_h and a new effective interaction $\mathcal{V}^{(h)}$ replacing Z_h, σ_h, μ_h and $\mathcal{V}^{(h)}$ respectively. As a consequence, a new fundamental problem must be faced: the size of these parameters and of the new effective interaction must be controlled, and in particular it must be proven that the weight of the local quartic term in $\mathcal{V}^{(h)}$ remains small under the iterations. This is not trivial at all, and in fact one of the major difficulties of the problem is in finding a suitable definition of the new parameters after each integration step. It will in fact become clear that there is some arbitrariness in their definition and the choice must be done with care, so that the flow of the effective coupling constants can be controlled.

In the present Chapter we will first describe the iterative procedure, including the definition of *localization*, crucial for the definition of the effective coupling constants. In the present Chapter we shall describe only the regime in which the effective parameters σ_h, μ_h are small; we shall call this regime the *anomalous* one, because σ_h, μ_h grow exponentially in this regime, with an exponent that is a non trivial function of λ . We then describe the result of the iteration in this regime, that is the bounds the kernels of the effective interaction satisfy at each step, *under the assumption that the size of the effective local quartic term remain small*. This key property (also called *vanishing of the Beta function*, for reasons that will become clear later) will be proven in next Chapter. The subsequent regime (in which σ_h, μ_h are of the same order of the mass of the field) must be studied with a different iterative procedure, and will be done in Chapter 8.

5.1. Multiscale analysis.

From the bound on $\det A_\psi^{(1)}(\mathbf{k})$ described in Theorem 4.1, we see that the ψ fields have a mass given by $\min\{|\sigma_1 - \mu_1|, |\sigma_1 + \mu_1|\}$, which can be arbitrarily small; their integration in the infrared region (small \mathbf{k}) needs a multiscale analysis. We introduce a *scaling parameter* $\gamma > 1$ which will be used to define a geometrically growing sequence of length scales $1, \gamma, \gamma^2, \dots$, *i.e.* of geometrically decreasing momentum scales γ^h , $h = 0, -1, -2, \dots$. Correspondingly we introduce C^∞ compact support functions $f_h(\mathbf{k})$ $h \leq 1$, with the following properties: if $|\mathbf{k}| \stackrel{\text{def}}{=} \sqrt{\sin^2 k + \sin^2 k_0}$, when $h \leq 0$, $f_h(\mathbf{k}) = 0$ for $|\mathbf{k}| < \gamma^{h-2}$ or $|\mathbf{k}| > \gamma^h$, and $f_h(\mathbf{k}) = 1$, if $|\mathbf{k}| = \gamma^{h-1}$; $f_1(\mathbf{k}) = 0$ for $|\mathbf{k}| \leq \gamma^{-1}$ and $f_1(\mathbf{k}) = 1$ for $|\mathbf{k}| \geq 1$; furthermore:

$$1 = \sum_{h=h_M}^1 f_h(\mathbf{k}) \quad , \quad \text{where :} \quad h_M = \min\{h : \gamma^h > \sqrt{2} \sin \frac{\pi}{M}\} \quad , \quad (5.1)$$

and $\sqrt{2} \sin(\pi/M)$ is the smallest momentum allowed by the antiperiodic boundary conditions, *i.e.* it is equal to $\min_{\mathbf{k} \in D_{-, -}} |\mathbf{k}|$.

The purpose is to perform the integration of (2.41) over the fermion fields in an iterative way. After each iteration we shall be left with a “simpler” Grassmannian integration to perform: if $h = 1, 0, -1, \dots, h_M$, we shall write

$$\Xi_{AT}^- = \int P_{Z_h, \sigma_h, \mu_h, C_h}(d\psi^{(\leq h)}) e^{-\mathcal{V}^{(h)}(\sqrt{Z_h} \psi^{(\leq h)}) - M^2 E_h} \quad , \quad \mathcal{V}^{(h)}(0) = 0 \quad , \quad (5.2)$$

where the quantities $Z_h, \sigma_h, \mu_h, C_h, P_{Z_h, \sigma_h, \mu_h, C_h}(d\psi^{(\leq h)})$, $\mathcal{V}^{(h)}$ and E_h have to be defined recursively and the result of the last iteration will be $\Xi_{AT}^- = e^{-M^2 E_{-1+h_M}}$, *i.e.* the value of the partition function.

$P_{Z_h, \sigma_h, \mu_h, C_h}(d\psi^{(\leq h)})$ is defined as

$$\begin{aligned}
 P_{Z_h, \sigma_h, \mu_h, C_h}(d\psi^{(\leq h)}) &= \\
 &= \mathcal{N}_h^{-1} \prod_{\mathbf{k} \in D_{-, -}} \prod_{\omega = \pm 1} d\psi_{\mathbf{k}, \omega}^{+(\leq h)} d\psi_{\mathbf{k}, \omega}^{-(\leq h)} \exp \left[-\frac{1}{4M^2} \sum_{\substack{\mathbf{k} \in D_{-, -} \\ C_h^{-1}(\mathbf{k}) > 0}} Z_h C_h(\mathbf{k}) \Psi_{\mathbf{k}}^{+(\leq h), T} A_{\psi}^{(h)}(\mathbf{k}) \Psi_{\mathbf{k}}^{(\leq h)} \right], \\
 A_{\psi}^{(h)}(\mathbf{k}) &= \begin{pmatrix} M^{(h)}(\mathbf{k}) & N^{(h)}(\mathbf{k}) \\ N^{(h)}(\mathbf{k}) & M^{(h)}(\mathbf{k}) \end{pmatrix} \\
 M^{(h)}(\mathbf{k}) &= \begin{pmatrix} i \sin k + \sin k_0 + a_h^+(\mathbf{k}) & -i(\sigma_h(\mathbf{k}) + c_h(\mathbf{k})) \\ i(\sigma_h(\mathbf{k}) + c_h(\mathbf{k})) & i \sin k - \sin k_0 + a_h^-(\mathbf{k}) \end{pmatrix} \\
 N^{(h)}(\mathbf{k}) &= \begin{pmatrix} b_h^+(\mathbf{k}) & i(\mu_h(\mathbf{k}) + d_h(\mathbf{k})) \\ -i(\mu_h(\mathbf{k}) + d_h(\mathbf{k})) & b_h^-(\mathbf{k}) \end{pmatrix},
 \end{aligned} \tag{5.3}$$

and

$$\Psi_{\mathbf{k}}^{+(\leq h), T} = (\hat{\psi}_{1, \mathbf{k}}^{+(\leq h)}, \hat{\psi}_{-1, \mathbf{k}}^{+(\leq h)}, \hat{\psi}_{1, -\mathbf{k}}^{-(\leq h)}, \hat{\psi}_{-1, -\mathbf{k}}^{-(\leq h)}) \quad \Psi_{\mathbf{k}}^{(\leq h), T} = (\hat{\psi}_{1, \mathbf{k}}^{-(\leq h)}, \hat{\psi}_{-1, \mathbf{k}}^{-(\leq h)}, \hat{\psi}_{1, -\mathbf{k}}^{+(\leq h)}, \hat{\psi}_{-1, -\mathbf{k}}^{+(\leq h)}), \tag{5.4}$$

\mathcal{N}_h is such that $\int P_{Z_h, \sigma_h, \mu_h, C_h}(d\psi^{(\leq h)}) = 1$, $C_h(\mathbf{k})^{-1} = \sum_{j=h_M}^h f_j(\mathbf{k})$. Moreover

$$\begin{aligned}
 \mathcal{V}^{(h)}(\psi) &= \sum_{n=1}^{\infty} \frac{1}{M^{2n}} \sum_{\mathbf{k}_1, \dots, \mathbf{k}_{2n-1}} \prod_{i=1}^{2n} \hat{\psi}_{\omega_i, \mathbf{k}_i}^{\alpha_i(\leq h)} \widehat{W}_{2n, \underline{\alpha}, \underline{\omega}}^{(h)}(\mathbf{k}_1, \dots, \mathbf{k}_{2n-1}) \delta \left(\sum_{i=1}^{2n} \alpha_i \mathbf{k}_i \right) \stackrel{def}{=} \\
 &\stackrel{def}{=} \sum_{n=1}^{\infty} \sum_{\substack{\mathbf{x}_1, \dots, \mathbf{x}_{2n} \\ \underline{\sigma}, \underline{j}, \underline{\omega}, \underline{\alpha}}} \prod_{i=1}^{2n} \partial_{j_i}^{\sigma_i} \psi_{\omega_i, \mathbf{x}_i}^{\alpha_i(\leq h)} W_{2n, \underline{\sigma}, \underline{j}, \underline{\alpha}, \underline{\omega}}^{(h)}(\mathbf{x}_1, \dots, \mathbf{x}_{2n}),
 \end{aligned} \tag{5.5}$$

where in the last line $j_i = 0, 1$, $\sigma_i \geq 0$ and ∂_{j_i} is the forward discrete derivative in the \hat{e}_j direction.

Note that the field $\psi^{(\leq h)}$, whose propagator is given by the inverse of $Z_h C_h(\mathbf{k}) A_{\psi}^{(h)}$, has the same support of $C_h^{-1}(\mathbf{k})$, that is on a strip of width γ^h around the singularity $\mathbf{k} = \mathbf{0}$. The field $\psi^{(\leq 1)}$ coincides with the field ψ of previous section, so that (4.3) is the same as (5.2) with $h = 1$.

It is crucial for the following to think $\widehat{W}_{2n, \underline{\alpha}, \underline{\omega}}^{(h)}$, $h \leq 1$, as functions of the variables $\sigma_k(\mathbf{k}), \mu_k(\mathbf{k})$, $k = h, h+1, \dots, 0, 1$, $\mathbf{k} \in D_{-, -}$. The iterative construction below will inductively imply that the dependence on these variables is well defined (note that for $h = 1$ we can think the kernels of $\mathcal{V}^{(1)}$ as functions of σ_1, μ_1 , see Theorem 4.1).

5.2. The localization operator.

We now begin to describe the iterative construction leading to (5.3). The first step consists in defining a *localization* operator \mathcal{L} acting on the kernels of $\mathcal{V}^{(h)}$, in terms of which we shall rewrite $\mathcal{V}^{(h)} = \mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}$, where $\mathcal{R} = 1 - \mathcal{L}$. The iterative integration procedure will use such splitting, see §5.3 below.

\mathcal{L} will be non zero only if acting on a kernel $\widehat{W}_{2n, \underline{\alpha}, \underline{\omega}}^{(h)}$ with $n = 1, 2$. In this case \mathcal{L} will be the combination of four different operators: \mathcal{L}_j , $j = 0, 1$, whose effect on a function of \mathbf{k} will be essentially to extract the term of order j from its Taylor series in \mathbf{k} ; and \mathcal{P}_j , $j = 0, 1$, whose effect on a functional of the sequence $\sigma_h(\mathbf{k}), \mu_h(\mathbf{k}), \dots, \sigma_1, \mu_1$ will be essentially to extract the term of order j from its power series in $\sigma_h(\mathbf{k}), \mu_h(\mathbf{k}), \dots, \sigma_1, \mu_1$.

The action of \mathcal{L}_j , $j = 0, 1$, on the kernels $\widehat{W}_{2n, \underline{\alpha}, \underline{\omega}}^{(h)}(\mathbf{k}_1, \dots, \mathbf{k}_{2n})$ is defined as follows.

1) If $n = 1$,

$$\begin{aligned}
 \mathcal{L}_0 \widehat{W}_{2, \underline{\alpha}, \underline{\omega}}^{(h)}(\mathbf{k}, \alpha_1 \alpha_2 \mathbf{k}) &= \frac{1}{4} \sum_{\eta, \eta' = \pm 1} \widehat{W}_{2, \underline{\alpha}, \underline{\omega}}^{(h)}(\bar{\mathbf{k}}_{\eta\eta'}, \alpha_1 \alpha_2 \bar{\mathbf{k}}_{\eta\eta'}) \\
 \mathcal{L}_1 \widehat{W}_{2, \underline{\alpha}, \underline{\omega}}^{(h)}(\mathbf{k}, \alpha_1 \alpha_2 \mathbf{k}) &= \frac{1}{4} \sum_{\eta, \eta' = \pm 1} \widehat{W}_{2, \underline{\alpha}, \underline{\omega}}^{(h)}(\bar{\mathbf{k}}_{\eta\eta'}, \alpha_1 \alpha_2 \bar{\mathbf{k}}_{\eta\eta'}) \left[\eta \frac{\sin k}{\sin \frac{\pi}{M}} + \eta' \frac{\sin k_0}{\sin \frac{\pi}{M}} \right],
 \end{aligned} \tag{5.6}$$

where $\bar{\mathbf{k}}_{\eta\eta'} = (\eta \frac{\pi}{M}, \eta' \frac{\pi}{M})$ are the smallest momenta allowed by the antiperiodic boundary conditions.

2) If $n = 2$, $\mathcal{L}_1 \widehat{W}_{4,\underline{\alpha},\underline{\omega}}^{(h)} = 0$ and

$$\mathcal{L}_0 \widehat{W}_{4,\underline{\alpha},\underline{\omega}}^{(h)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \stackrel{def}{=} \widehat{W}_{4,\underline{\alpha},\underline{\omega}}^{(h)}(\bar{\mathbf{k}}_{++}, \bar{\mathbf{k}}_{++}, \bar{\mathbf{k}}_{++}, \bar{\mathbf{k}}_{++}) . \quad (5.7)$$

3) If $n > 2$, $\mathcal{L}_0 \widehat{W}_{2n,\underline{\alpha},\underline{\omega}} = \mathcal{L}_1 \widehat{W}_{2n,\underline{\alpha},\underline{\omega}} = 0$.

The action of \mathcal{P}_j , $j = 0, 1$, on the kernels $\widehat{W}_{2n,\underline{\alpha},\underline{\omega}}$, thought as functionals of the sequence $\sigma_h(\mathbf{k}), \mu_h(\mathbf{k}), \dots, \sigma_1, \mu_1$ is defined as follows.

$$\begin{aligned} \mathcal{P}_0 \widehat{W}_{2n,\underline{\alpha},\underline{\omega}} &\stackrel{def}{=} \widehat{W}_{2n,\underline{\alpha},\underline{\omega}} \Big|_{\underline{\sigma}^{(h)} = \underline{\mu}^{(h)} = 0} \\ \mathcal{P}_1 \widehat{W}_{2n,\underline{\alpha},\underline{\omega}} &\stackrel{def}{=} \sum_{k \geq h, \mathbf{k}} \left[\sigma_k(\mathbf{k}) \frac{\partial \widehat{W}_{2n,\underline{\alpha},\underline{\omega}}}{\partial \sigma_k(\mathbf{k})} \Big|_{\underline{\sigma}^{(h)} = \underline{\mu}^{(h)} = 0} + \mu_k(\mathbf{k}) \frac{\partial \widehat{W}_{2n,\underline{\alpha},\underline{\omega}}}{\partial \mu_k(\mathbf{k})} \Big|_{\underline{\sigma}^{(h)} = \underline{\mu}^{(h)} = 0} \right] . \end{aligned} \quad (5.8)$$

Given $\mathcal{L}_j, \mathcal{P}_j$, $j = 0, 1$ as above, we define the action of \mathcal{L} on the kernels $\widehat{W}_{2n,\underline{\alpha},\underline{\omega}}$ as follows.

1) If $n = 1$, then

$$\mathcal{L} \widehat{W}_{2,\underline{\alpha},\underline{\omega}} \stackrel{def}{=} \begin{cases} \mathcal{L}_0(\mathcal{P}_0 + \mathcal{P}_1) \widehat{W}_{2,\underline{\alpha},\underline{\omega}} & \text{if } \omega_1 + \omega_2 = 0 \text{ and } \alpha_1 + \alpha_2 = 0, \\ \mathcal{L}_0 \mathcal{P}_1 \widehat{W}_{2,\underline{\alpha},\underline{\omega}} & \text{if } \omega_1 + \omega_2 = 0 \text{ and } \alpha_1 + \alpha_2 \neq 0, \\ \mathcal{L}_1 \mathcal{P}_0 \widehat{W}_{2,\underline{\alpha},\underline{\omega}} & \text{if } \omega_1 + \omega_2 \neq 0 \text{ and } \alpha_1 + \alpha_2 = 0, \\ 0 & \text{if } \omega_1 + \omega_2 \neq 0 \text{ and } \alpha_1 + \alpha_2 \neq 0. \end{cases}$$

2) If $n = 2$, then $\mathcal{L} \widehat{W}_{4,\underline{\alpha},\underline{\omega}} \stackrel{def}{=} \mathcal{L}_0 \mathcal{P}_0 \widehat{W}_{4,\underline{\alpha},\underline{\omega}}$.

3) If $n > 2$, then $\mathcal{L} \widehat{W}_{2n,\underline{\alpha},\underline{\omega}} = 0$.

Finally, the effect of \mathcal{L} on $\mathcal{V}^{(h)}$ is, by definition, to replace on the r.h.s. of (4.8) $\widehat{W}_{2n,\underline{\alpha},\underline{\omega}}$ with $\mathcal{L} \widehat{W}_{2n,\underline{\alpha},\underline{\omega}}$. Note that $\mathcal{L}^2 \mathcal{V}^{(h)} = \mathcal{L} \mathcal{V}^{(h)}$.

Using the previous definitions we get the following result. We use the notation $\underline{\sigma}^{(h)} = \{\sigma_k(\mathbf{k})\}_{\mathbf{k} \in D_{-,-}^{k=h,\dots,1}}$ and $\underline{\mu}^{(h)} = \{\mu_k(\mathbf{k})\}_{\mathbf{k} \in D_{-,-}^{k=h,\dots,1}}$.

LEMMA 5.1. *Let the action of \mathcal{L} on $\mathcal{V}^{(h)}$ be defined as above. Then*

$$\mathcal{L} \mathcal{V}^{(h)}(\psi^{(\leq h)}) = (s_h + \gamma^h n_h) F_\sigma^{(\leq h)} + m_h F_\mu^{(\leq h)} + l_h F_\lambda^{(\leq h)} + z_h F_\zeta^{(\leq h)} , \quad (5.9)$$

where s_h, n_h, m_h, l_h and z_h are real constants and: s_h is linear in $\underline{\sigma}^{(h)}$ and independent of $\underline{\mu}^{(h)}$; m_h is linear in $\underline{\mu}^{(h)}$ and independent of $\underline{\sigma}^{(h)}$; n_h, l_h, z_h are independent of $\underline{\sigma}^{(h)}, \underline{\mu}^{(h)}$; moreover, if $D_h \stackrel{def}{=} D_{-,-} \cap \{\mathbf{k} : C_h^{-\top}(\mathbf{k}) > 0\}$,

$$\begin{aligned} F_\sigma^{(\leq h)}(\psi^{(\leq h)}) &= \frac{1}{2M^2} \sum_{\mathbf{k} \in D_h} \sum_{\omega=\pm 1} (-i\omega) \widehat{\psi}_{\omega,\mathbf{k}}^{+(\leq h)} \widehat{\psi}_{-\omega,\mathbf{k}}^{-(\leq h)} \stackrel{def}{=} \frac{1}{M^2} \sum_{\mathbf{k} \in D_h} \widehat{F}_\sigma^{(\leq h)}(\mathbf{k}) , \\ F_\mu^{(\leq h)}(\psi^{(\leq h)}) &= \frac{1}{4M^2} \sum_{\mathbf{k} \in D_h} \sum_{\alpha,\omega=\pm 1} i\omega \widehat{\psi}_{\omega,\mathbf{k}}^{\alpha(\leq h)} \widehat{\psi}_{-\omega,-\mathbf{k}}^{\alpha(\leq h)} \stackrel{def}{=} \frac{1}{M^2} \sum_{\mathbf{k} \in D_h} \widehat{F}_\mu^{(\leq h)}(\mathbf{k}) , \\ F_\lambda^{(\leq h)}(\psi^{(\leq h)}) &= \frac{1}{M^8} \sum_{\mathbf{k}_1, \dots, \mathbf{k}_4 \in D_h} \widehat{\psi}_{1,\mathbf{k}_1}^{+(\leq h)} \widehat{\psi}_{-1,\mathbf{k}_2}^{+(\leq h)} \widehat{\psi}_{-1,\mathbf{k}_3}^{-(\leq h)} \widehat{\psi}_{1,\mathbf{k}_4}^{-(\leq h)} \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \\ F_\zeta^{(\leq h)}(\psi^{(\leq h)}) &= \frac{1}{2M^2} \sum_{\mathbf{k} \in D_h} \sum_{\omega=\pm 1} (i \sin k + \omega \sin k_0) \widehat{\psi}_{\omega,\mathbf{k}}^{+(\leq h)} \widehat{\psi}_{\omega,\mathbf{k}}^{-(\leq h)} \stackrel{def}{=} \frac{1}{M^2} \sum_{\mathbf{k} \in D_h} \widehat{F}_\zeta^{(\leq h)}(\mathbf{k}) . \end{aligned} \quad (5.10)$$

where $\delta(\mathbf{k}) = M^2 \sum_{\mathbf{n} \in \mathbb{Z}^2} \delta_{\mathbf{k}, 2\pi\mathbf{n}}$.

Remark. The application of \mathcal{L} to the kernels of the effective potential generates the sum in (5.9), i.e. a linear combination of the Grassmannian monomials in (5.10) which, in the renormalization group language, are called “*relevant*” (the first two) or “*marginal*” operators (the two others).

PROOF OF LEMMA 5.1 Lemma 5.1 can be proven repeating the discussion in §4.3 above. Note in fact that the result of §4.3, as presented in Theorem 4.1, can be reformulated by saying that

$$\mathcal{LV}^{(1)}(\psi) = (s_1 + \gamma n_1) F_\sigma^{(\leq 1)} + m_1 F_\mu^{(\leq 1)} + l_1 F_\lambda^{(\leq 1)} + z_1 F_\zeta^{(\leq 1)}, \quad (5.11)$$

where s_1, n_1, m_1, l_1 and z_1 are real constants and: s_1 is linear in σ_1 and independent of μ_1 ; m_1 is linear in μ_1 and independent of σ_1 ; n_1, l_1, z_1 are independent of σ_1, μ_1 .

It is now sufficient to note that the symmetries (1)–(6) discussed in §4.3 are preserved by the iterative integration procedure: in fact it is easy to verify that $\mathcal{LV}^{(h)}$, $\mathcal{RV}^{(h)}$ and $P_{Z_{h-1}, \sigma_{h-1}, \mu_{h-1}, \tilde{f}_h}(d\psi^{(h)})$ are, step by step, separately invariant under the transformations (1)–(6). Then the same proof leading to (5.11) leads to (5.9) (it is sufficient to replace any scale label = 1 with h).

We now consider the operator $\mathcal{R} \stackrel{\text{def}}{=} 1 - \mathcal{L}$. The following result holds. We use the notation $\mathcal{R}_1 = 1 - \mathcal{L}_0$, $\mathcal{R}_2 = 1 - \mathcal{L}_0 - \mathcal{L}_1$, $\mathcal{S}_1 = 1 - \mathcal{P}_0$, $\mathcal{S}_2 = 1 - \mathcal{P}_0 - \mathcal{P}_1$.

LEMMA 5.2. *The action of \mathcal{R} on $\widehat{W}_{2n, \underline{\alpha}, \underline{\omega}}$ for $n = 1, 2$ is the following.*

1) If $n = 1$, then

$$\mathcal{R}\widehat{W}_{2, \underline{\alpha}, \underline{\omega}} = \begin{cases} [\mathcal{S}_2 + \mathcal{R}_2(\mathcal{P}_0 + \mathcal{P}_1)]\widehat{W}_{2, \underline{\alpha}, \underline{\omega}} & \text{if } \omega_1 + \omega_2 = 0, \\ [\mathcal{R}_1\mathcal{S}_1 + \mathcal{R}_2\mathcal{P}_0]\widehat{W}_{2, \underline{\alpha}, \underline{\omega}} & \text{if } \omega_1 + \omega_2 \neq 0 \text{ and } \alpha_1 + \alpha_2 = 0, \\ \mathcal{R}_1\mathcal{S}_1\widehat{W}_{2, \underline{\alpha}, \underline{\omega}} & \text{if } \omega_1 + \omega_2 \neq 0 \text{ and } \alpha_1 + \alpha_2 \neq 0, \end{cases}$$

2) If $n = 2$, then $\mathcal{R}\widehat{W}_{4, \underline{\alpha}, \underline{\omega}} = [\mathcal{S}_1 + \mathcal{R}_1\mathcal{P}_0]\widehat{W}_{4, \underline{\alpha}, \underline{\omega}}$.

Remark. The effect of \mathcal{R}_j , $j = 1, 2$ on $\widehat{W}_{2n, \underline{\alpha}, \underline{\omega}}^{(h)}$ consists in extracting the rest of a Taylor series in \mathbf{k} of order j . The effect of \mathcal{S}_j , $j = 1, 2$ on $\widehat{W}_{2n, \underline{\alpha}, \underline{\omega}}^{(h)}$ consists in extracting the rest of a power series in $(\underline{\sigma}^{(h)}, \underline{\mu}^{(h)})$ of order j . The definitions are given in such a way that $\mathcal{R}\widehat{W}_{2n, \underline{\alpha}, \underline{\omega}}$ is at least quadratic in $\mathbf{k}, \underline{\sigma}^{(h)}, \underline{\mu}^{(h)}$ if $n = 1$ and at least linear in $\mathbf{k}, \underline{\sigma}^{(h)}, \underline{\mu}^{(h)}$ when $n = 2$. This will give dimensional gain factors in the bounds for $\mathcal{R}\widehat{W}_{2n, \underline{\alpha}, \underline{\omega}}^{(h)}$ w.r.t. the bounds for $\widehat{W}_{2n, \underline{\alpha}, \underline{\omega}}^{(h)}$, $n = 1, 2$, as we shall see in details in §5.5.

PROOF OF LEMMA 5.2 It is sufficient to note that the symmetry properties discussed in §4.3 imply that: $\mathcal{L}_1 W_{2, \underline{\alpha}, \underline{\omega}} = 0$ if $\omega_1 + \omega_2 = 0$; $\mathcal{L}_0 W_{2, \underline{\alpha}, \underline{\omega}} = 0$ if $\omega_1 + \omega_2 \neq 0$; $\mathcal{P}_0 W_{2, \underline{\alpha}, \underline{\omega}} = 0$ if $\alpha_1 + \alpha_2 \neq 0$; and use the definitions of \mathcal{R}_i , \mathcal{S}_i , $i = 1, 2$.

5.3. Renormalization.

Once that the above definitions are given we can describe our integration procedure for $h \leq 0$. We start from (5.2) and we rewrite it as

$$\int P_{Z_h, \sigma_h, \mu_h, C_h}(d\psi^{(\leq h)}) e^{-\mathcal{LV}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) - \mathcal{RV}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) - M^2 E_h}, \quad (5.12)$$

with $\mathcal{LV}^{(h)}$ as in (5.9). Then we include the quadratic part of $\mathcal{LV}^{(h)}$ (except the term proportional to n_h) in the fermionic integration, so obtaining

$$\int P_{\hat{Z}_{h-1}, \sigma_{h-1}, \mu_{h-1}, C_h}(d\psi^{(\leq h)}) e^{-l_h F_\lambda(\sqrt{Z_h}\psi^{(\leq h)}) - \gamma^h n_h F_\sigma(\sqrt{Z_h}\psi^{(\leq h)}) - \mathcal{RV}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) - M^2 E_h}, \quad (5.13)$$

where $\widehat{Z}_{h-1}(\mathbf{k}) \stackrel{\text{def}}{=} Z_h(1 + z_h C_h^{-1}(\mathbf{k}))$ and

$$\begin{aligned} \sigma_{h-1}(\mathbf{k}) &\stackrel{\text{def}}{=} \frac{Z_h}{\widehat{Z}_{h-1}(\mathbf{k})} (\sigma_h(\mathbf{k}) + s_h C_h^{-1}(\mathbf{k})) \quad , \quad \mu_{h-1}(\mathbf{k}) \stackrel{\text{def}}{=} \frac{Z_h}{\widehat{Z}_{h-1}(\mathbf{k})} (\mu_h(\mathbf{k}) + m_h C_h^{-1}(\mathbf{k})) \\ a_{h-1}^\omega(\mathbf{k}) &\stackrel{\text{def}}{=} \frac{Z_h}{\widehat{Z}_{h-1}(\mathbf{k})} a_h^\omega(\mathbf{k}) \quad , \quad b_{h-1}^\omega(\mathbf{k}) \stackrel{\text{def}}{=} \frac{Z_h}{\widehat{Z}_{h-1}(\mathbf{k})} b_h^\omega(\mathbf{k}) \\ c_{h-1}(\mathbf{k}) &\stackrel{\text{def}}{=} \frac{Z_h}{\widehat{Z}_{h-1}(\mathbf{k})} c_h(\mathbf{k}) \quad , \quad d_{h-1}(\mathbf{k}) \stackrel{\text{def}}{=} \frac{Z_h}{\widehat{Z}_{h-1}(\mathbf{k})} d_h(\mathbf{k}) . \end{aligned} \quad (5.14)$$

The integration in (5.13) differs from the one in (5.2) and (5.12): $P_{\widehat{Z}_{h-1}, \sigma_{h-1}, \mu_{h-1}, C_h}$ is defined by (5.3) with Z_h and $A_\psi^{(h)}$ replaced by $\widehat{Z}_{h-1}(\mathbf{k})$ and $A_\psi^{(h-1)}$.

Now we can perform the integration of the $\psi^{(h)}$ field. It is convenient to rescale the fields:

$$\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}) \stackrel{\text{def}}{=} \lambda_h F_\lambda(\sqrt{Z_{h-1}}\psi^{(\leq h)}) + \gamma^h \nu_h F_\sigma(\sqrt{Z_{h-1}}\psi^{(\leq h)}) + \mathcal{R}\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}) , \quad (5.15)$$

where

$$\lambda_h = \left(\frac{Z_h}{Z_{h-1}}\right)^2 l_h , \quad \nu_h = \frac{Z_h}{Z_{h-1}} n_h , \quad (5.16)$$

and $\mathcal{R}\mathcal{V}^{(h)} = (1 - \mathcal{L})\mathcal{V}^{(h)}$ is the irrelevant part of $\mathcal{V}^{(h)}$, and rewrite (5.13) as

$$e^{-M^2(t_h + E_h)} \int P_{Z_{h-1}, \sigma_{h-1}, \mu_{h-1}, C_{h-1}}(d\psi^{(\leq h-1)}) \int P_{Z_{h-1}, \sigma_{h-1}, \mu_{h-1}, \widetilde{f}_h^{-1}}(d\psi^{(h)}) e^{-\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)})} \quad (5.17)$$

where we used the decomposition $\psi^{(\leq h)} = \psi^{(\leq h-1)} + \psi^{(h)}$ (and $\psi^{(\leq h-1)}, \psi^{(h)}$ are independent) and $\widetilde{f}_h(\mathbf{k})$ is defined by the relation $C_h^{-1}(\mathbf{k})\widehat{Z}_{h-1}^{-1}(\mathbf{k}) = C_{h-1}^{-1}(\mathbf{k})Z_{h-1}^{-1} + \widetilde{f}_h(\mathbf{k})Z_{h-1}^{-1}$, namely:

$$\widetilde{f}_h(\mathbf{k}) \stackrel{\text{def}}{=} Z_{h-1} \left[\frac{C_h^{-1}(\mathbf{k})}{\widehat{Z}_{h-1}(\mathbf{k})} - \frac{C_{h-1}^{-1}(\mathbf{k})}{Z_{h-1}} \right] = f_h(\mathbf{k}) \left[1 + \frac{z_h f_{h+1}(\mathbf{k})}{1 + z_h f_h(\mathbf{k})} \right] . \quad (5.18)$$

Note that $\widetilde{f}_h(\mathbf{k})$ has the same support as $f_h(\mathbf{k})$. Moreover $P_{Z_{h-1}, \sigma_{h-1}, \mu_{h-1}, \widetilde{f}_h^{-1}}(d\psi^{(h)})$ is defined in the same way as $P_{\widehat{Z}_{h-1}, \sigma_{h-1}, \mu_{h-1}, C_h}(d\psi^{(h)})$, with $\widehat{Z}_{h-1}(\mathbf{k})$ resp. C_h replaced by Z_{h-1} resp. \widetilde{f}_h^{-1} . The *single scale* propagator is

$$\int P_{Z_{h-1}, \sigma_{h-1}, \mu_{h-1}, \widetilde{f}_h^{-1}}(d\psi^{(h)}) \psi_{\mathbf{x}, \omega}^{\alpha(h)} \psi_{\mathbf{y}, \omega'}^{\alpha'(h)} = \frac{1}{Z_{h-1}} g_{\underline{a}, \underline{a}'}^{(h)}(\mathbf{x} - \mathbf{y}) \quad , \quad \underline{a} = (\alpha, \omega) \quad , \quad \underline{a}' = (\alpha', \omega') , \quad (5.19)$$

where

$$g_{\underline{a}, \underline{a}'}^{(h)}(\mathbf{x} - \mathbf{y}) = \frac{1}{2M^2} \sum_{\mathbf{k}} e^{i\alpha\alpha'\mathbf{k}(\mathbf{x}-\mathbf{y})} \widetilde{f}_h(\mathbf{k}) [A_\psi^{(h-1)}(\mathbf{k})]_{j(\underline{a}), j'(\underline{a}')}^{-1} \quad (5.20)$$

with $j(-, 1) = j'(+, 1) = 1$, $j(-, -1) = j'(+, -1) = 2$, $j(+, 1) = j'(-, 1) = 3$ and $j(+, -1) = j'(-, -1) = 4$. One finds that $g_{\underline{a}, \underline{a}'}^{(h)}(\mathbf{x}) = g_{\omega, \omega'}^{(1, h)}(\mathbf{x}) - \alpha\alpha' g_{\omega, \omega'}^{(2, h)}(\mathbf{x})$, where $g_{\omega, \omega'}^{(j, h)}(\mathbf{x})$, $j = 1, 2$ are defined in Appendix A4.

The long distance behaviour of the propagator is given by the following Lemma, proved in Appendix A4.

LEMMA 5.3. *Let $\sigma_h \stackrel{\text{def}}{=} \sigma_h(\mathbf{0})$ and $\mu_h \stackrel{\text{def}}{=} \mu_h(\mathbf{0})$ and assume $|\lambda| \leq \varepsilon_1$ for a small constant ε_1 . Suppose that for $h > \bar{h}$*

$$|z_h| \leq \frac{1}{2} \quad , \quad |s_h| \leq \frac{1}{2} |\sigma_h| \quad , \quad |m_h| \leq \frac{1}{2} |\mu_h| , \quad (5.21)$$

that there exists c s.t.

$$e^{-c|\lambda|} \leq \left| \frac{\sigma_h}{\sigma_{h-1}} \right| \leq e^{c|\lambda|} \quad , \quad e^{-c|\lambda|} \leq \left| \frac{\mu_h}{\mu_{h-1}} \right| \leq e^{c|\lambda|} \quad , \quad e^{-c|\lambda|^2} \leq \left| \frac{Z_h}{Z_{h-1}} \right| \leq e^{c|\lambda|^2} , \quad (5.22)$$

and that, for some constant C_1 ,

$$\frac{|\sigma_{\bar{h}}|}{\gamma^{\bar{h}}} \leq C_1 \quad , \quad \frac{|\mu_{\bar{h}}|}{\gamma^{\bar{h}}} \leq C_1 ; \quad (5.23)$$

then, for all $h \geq \bar{h}$, given the positive integers N, n_0, n_1 and putting $n = n_0 + n_1$, there exists a constant $C_{N,n}$ s.t.

$$|\partial_{x_0}^{n_0} \partial_x^{n_1} g_{\underline{a}, \underline{a}'}^{(h)}(\mathbf{x} - \mathbf{y})| \leq C_{N,n} \frac{\gamma^{(1+n)h}}{1 + (\gamma^h |\mathbf{d}(\mathbf{x} - \mathbf{y})|)^N} \quad , \quad \text{where} \quad \mathbf{d}(\mathbf{x}) = \frac{M}{\pi} \left(\sin \frac{\pi x}{M}, \sin \frac{\pi x_0}{M} \right). \quad (5.24)$$

Furthermore, if $\mathcal{P}_0, \mathcal{P}_1$ are defined as in (5.8) and $\mathcal{S}_1, \mathcal{S}_2$ are defined as in Lemma 5.2, we have that $\mathcal{P}_j g_{\underline{a}, \underline{a}'}^{(h)}$, $j = 0, 1$ and $\mathcal{S}_j g_{\underline{a}, \underline{a}'}^{(h)}$, $j = 1, 2$, satisfy the same bound (5.24), times a factor $\left(\frac{|\sigma_h| + |\mu_h|}{\gamma^h} \right)^j$. The bounds for $\mathcal{P}_0 g_{\underline{a}, \underline{a}'}^{(h)}$ and $\mathcal{P}_1 g_{\underline{a}, \underline{a}'}^{(h)}$ hold even without hypothesis (5.23).

After the integration of the field on scale h we are left with an integral involving the fields $\psi^{(\leq h-1)}$ and the new effective interaction $\mathcal{V}^{(h-1)}$, defined as

$$e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leq h-1)}) - \tilde{E}_h M^2} = \int P_{Z_{h-1}, \sigma_{h-1}, \mu_{h-1}, \tilde{f}_h} (d\psi^{(h)}) e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)})}. \quad (5.25)$$

It is easy to see that $\mathcal{V}^{(h-1)}$ is of the form (5.5) and that $E_{h-1} = E_h + t_h + \tilde{E}_h$. It is sufficient to use the well known identity

$$M^2 \tilde{E}_h + \mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leq h-1)}) = \sum_{n \geq 1} \frac{1}{n!} (-1)^{n+1} \mathcal{E}_h^T(\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}); n), \quad (5.26)$$

where $\mathcal{E}_h^T(X(\psi^{(h)}); n)$ is the truncated expectation of order n w.r.t. the propagator $Z_{h-1}^{-1} g_{\underline{a}, \underline{a}'}^{(h)}$, defined as

$$\mathcal{E}_h^T(X(\psi^{(h)}); n) = \frac{\partial}{\partial \lambda^n} \log \int P_{Z_{h-1}, \sigma_{h-1}, \mu_{h-1}, \tilde{f}_h} (d\psi^{(h)}) e^{\lambda X(\psi^{(h)})} \Big|_{\lambda=0}. \quad (5.27)$$

Note that the above procedure allow us to write the *running coupling constants* $\vec{v}_{h-1} = (\lambda_{h-1}, \nu_{h-1})$, $h \leq 1$, in terms of \vec{v}_k , $h \leq k \leq 1$, namely

$$\vec{v}_{h-1} = \beta_h(\vec{v}_h, \dots, \vec{v}_1), \quad (5.28)$$

where β_h is the so-called *Beta function*.

5.4. Analiticity of the effective potential

We have expressed the effective potential $\mathcal{V}^{(h)}$ in terms of the *running coupling constants* λ_k, ν_k , $k \geq h$, and of the *renormalization constants* $Z_k, \mu_k(\mathbf{k}), \sigma_k(\mathbf{k})$, $k \geq h$.

In next section we will prove the following result.

THEOREM 5.1. *Let $\sigma_h \stackrel{\text{def}}{=} \sigma_h(\mathbf{0})$ and $\mu_h \stackrel{\text{def}}{=} \mu_h(\mathbf{0})$ and assume $|\lambda| \leq \varepsilon_1$ for a small constant ε_1 . Suppose that for $h > \bar{h}$ the hypothesis (5.21), (5.22) and (5.23) hold. If, for some constant c ,*

$$\max_{h > \bar{h}} \{|\lambda_h|, |\nu_h|\} \leq c|\lambda|, \quad (5.29)$$

then there exists $C > 0$ s.t. the kernels in (5.5) satisfy

$$\int d\mathbf{x}_1 \cdots d\mathbf{x}_{2n} |W_{2n, \underline{\sigma}, \underline{j}, \underline{a}, \underline{a}'}^{(\bar{h})}(\mathbf{x}_1, \dots, \mathbf{x}_{2n})| \leq M^2 \gamma^{-\bar{h} D_k(n)} (C |\lambda|)^{\max(1, n-1)} \quad (5.30)$$

where $D_k(n) = -2 + n + k$ and $k = \sum_{i=1}^{2n} \sigma_i$.

Moreover $|\tilde{E}_{\bar{h}+1}| + |t_{\bar{h}+1}| \leq c|\lambda|\gamma^{2\bar{h}}$ and the kernels of $\mathcal{LV}^{(\bar{h})}$ satisfy

$$|s_{\bar{h}}| \leq C|\lambda||\sigma_{\bar{h}}| \quad , \quad |m_{\bar{h}}| \leq C|\lambda||\mu_{\bar{h}}| \quad (5.31)$$

and

$$|n_{\bar{h}}| \leq C|\lambda| \quad , \quad |z_{\bar{h}}| \leq C|\lambda|^2 \quad , \quad |l_{\bar{h}}| \leq C|\lambda|^2 . \quad (5.32)$$

The bounds (5.31) holds even if (5.23) does not hold. The bounds (5.32) holds even if (5.23) and the first two of (5.22) do not hold.

Remarks.

- 1) The above result immediately implies analyticity of the effective potential of scale h in the running coupling constants λ_k, ν_k , $k \geq h$, under the assumptions (5.21), (5.22), (5.23) and (5.29).
- 2) The assumptions (5.22) and (5.29) will be proved in next Chapter, solving the *flow equations* for $\vec{v}_h = (\lambda_h, \nu_h)$ and Z_h, σ_h, μ_h , given by $\vec{v}_{h-1} = \beta_h(\vec{v}_h, \dots, \vec{v}_1)$, $Z_{h-1} = Z_h(1 + z_h)$ and (5.14). They will be proved to be true up to $h = -\infty$.

5.5. Proof of Theorem 5.1.

It is possible to write $\mathcal{V}^{(h)}$ (5.5) in terms of *Gallavotti–Nicolò’ trees*. The detailed derivation of this representation can be found in the reviews papers [G1][GM] and in my diploma thesis [G]. We do not repeat here the details, we only give the basic definitions, in order to make the subsequent discussion self consistent.

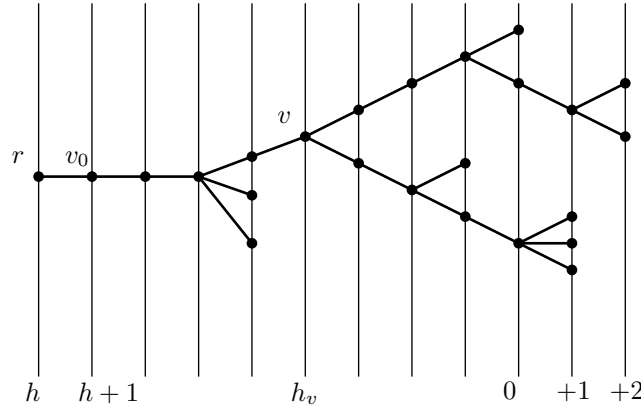


FIG. 5. A tree with its scale labels.

Let us introduce the following definitions and notations.

- 1) Let us consider the family of all trees which can be constructed by joining a point r , the *root*, with an ordered set of $n \geq 1$ points, the *endpoints* of the *unlabeled tree*, so that r is not a branching point. n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. Then the number of unlabeled trees with n end-points is bounded by 4^n .

- 2) We associate a label $h \leq 0$ with the root and we denote $\mathcal{T}_{h,n}$ the corresponding set of labeled trees with n endpoints. Moreover, we introduce a family of vertical lines, labeled by an integer taking values in $[h, 2]$,

and we represent any tree $\tau \in \mathcal{T}_{h,n}$ so that, if v is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > h$, to be called the *scale* of v , while the root is on the line with index h . There is the constraint that, if v is an endpoint, $h_v > h + 1$; if there is only one end-point its scale must be equal to $h + 2$, for $h \leq 0$. Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and can not be an endpoint; its scale is $h + 1$.

3) With each endpoint v of scale $h_v = +2$ we associate one of the contributions to $\mathcal{V}^{(1)}$ given by (4.5); with each endpoint v of scale $h_v \leq 1$ one of the terms in $\mathcal{L}\mathcal{V}^{(h_v-1)}$ defined in (5.9). Moreover, we impose the constraint that, if v is an endpoint and $h_v \leq 1$, $h_v = h_{v'} + 1$, if v' is the non trivial vertex immediately preceding v .

4) We introduce a *field label* f to distinguish the field variables appearing in the terms associated with the endpoints as in item 3); the set of field labels associated with the endpoint v will be called I_v . Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v ; $\mathbf{x}(f)$, $\sigma(f)$ and $\omega(f)$ will denote the space-time point, the σ index and the ω index, respectively, of the field variable with label f .

5) We associate with any vertex v of the tree a subset P_v of I_v , the *external fields* of v . These subsets must satisfy various constraints. First of all, if v is not an endpoint and v_1, \dots, v_{s_v} are the s_v vertices immediately following it, then $P_v \subset \cup_i P_{v_i}$; if v is an endpoint, $P_v = I_v$. We shall denote Q_{v_i} the intersection of P_v and P_{v_i} ; this definition implies that $P_v = \cup_i Q_{v_i}$. The subsets $P_{v_i} \setminus Q_{v_i}$, whose union will be made, by definition, of the *internal fields* of v , have to be non empty, if $s_v > 1$, that is if v is a non trivial vertex. Given $\tau \in \mathcal{T}_{j,n}$, there are many possible choices of the subsets P_v , $v \in \tau$, compatible with the previous constraints; let us call \mathbf{P} one of this choices. Given \mathbf{P} , we consider the family $\mathcal{G}_{\mathbf{P}}$ of all connected Feynman graphs, such that, for any $v \in \tau$, the internal fields of v are paired by propagators of scale h_v , so that the following condition is satisfied: for any $v \in \tau$, the subgraph built by the propagators associated with all vertices $v' \geq v$ is connected. The sets P_v have, in this picture, the role of the external legs of the subgraph associated with v . The graphs belonging to $\mathcal{G}_{\mathbf{P}}$ will be called *compatible with \mathbf{P}* and we shall denote \mathcal{P}_{τ} the family of all choices of \mathbf{P} such that $\mathcal{G}_{\mathbf{P}}$ is not empty.

6) we associate with any vertex v an index $\rho_v \in \{s, p\}$ and correspondingly an operator \mathcal{R}_{ρ_v} , where \mathcal{R}_s or \mathcal{R}_p are defined as

$$\mathcal{R}_s \stackrel{\text{def}}{=} \begin{cases} \mathcal{S}_2 & \text{if } n = 1 \text{ and } \omega_1 + \omega_2 = 0, \\ \mathcal{R}_1 \mathcal{S}_1 & \text{if } n = 1 \text{ and } \omega_1 + \omega_2 \neq 0, \\ \mathcal{S}_1 & \text{if } n = 2, \\ 1 & \text{if } n > 2; \end{cases} \quad (5.33)$$

and

$$\mathcal{R}_p \stackrel{\text{def}}{=} \begin{cases} \mathcal{R}_2(\mathcal{P}_0 + \mathcal{P}_1) & \text{if } n = 1 \text{ and } \omega_1 + \omega_2 = 0, \\ \mathcal{R}_2 \mathcal{P}_0 & \text{if } n = 1, \omega_1 + \omega_2 \neq 0 \text{ and } \alpha_1 + \alpha_2 = 0, \\ 0 & \text{if } n = 1, \omega_1 + \omega_2 \neq 0 \text{ and } \alpha_1 + \alpha_2 \neq 0, \\ \mathcal{R}_1 \mathcal{P}_0 & \text{if } n = 2, \\ 0 & \text{if } n > 2. \end{cases} \quad (5.34)$$

Note that $\mathcal{R}_s + \mathcal{R}_p = \mathcal{R}$, see Lemma 5.2.

The effective potential can be written in the following way:

$$\mathcal{V}^{(h)}(\sqrt{Z_h} \psi^{(\leq h)}) + M^2 \tilde{E}_{h+1} = \sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{h,n}} \mathcal{V}^{(h)}(\tau, \sqrt{Z_h} \psi^{(\leq h)}), \quad (5.35)$$

where, if v_0 is the first vertex of τ and τ_1, \dots, τ_s are the subtrees of τ with root v_0 , $\mathcal{V}^{(h)}(\tau, \sqrt{Z_h} \psi^{(\leq h)})$ is defined inductively by the relation

$$\begin{aligned} \mathcal{V}^{(h)}(\tau, \sqrt{Z_h} \psi^{(\leq h)}) = \\ \frac{(-1)^{s+1}}{s!} \mathcal{E}_{h+1}^T [\bar{V}^{(h+1)}(\tau_1, \sqrt{Z_h} \psi^{(\leq h+1)}); \dots; \bar{V}^{(h+1)}(\tau_s, \sqrt{Z_h} \psi^{(\leq h+1)})] , \end{aligned} \quad (5.36)$$

5.6 UNIVERSALITY AND NON-UNIVERSALITY IN THE ASHKIN–TELLER MODEL

and $\bar{V}^{(h+1)}(\tau_i, \sqrt{Z_h}\psi^{(\leq h+1)})$:

- a) is equal to $\mathcal{R}_{\rho_{v_i}}\hat{\mathcal{V}}^{(h+1)}(\tau_i, \sqrt{Z_h}\psi^{(\leq h+1)})$ if the subtree τ_i with first vertex v_i is not trivial (see (5.15) for the definition of $\hat{\mathcal{V}}^{(h)}$);
- b) if τ_i is trivial and $h \leq -1$, it is equal to one of the terms in $\mathcal{L}\hat{\mathcal{V}}^{(h+1)}$, see (5.15), or, if $h = 0$, to one of the terms contributing to $\hat{\mathcal{V}}^{(1)}(\sqrt{Z_1}\psi^{\leq 1})$.

5.6. The explicit expression for the kernels of $\mathcal{V}^{(h)}$ can be found from (5.35) and (5.36) by writing the truncated expectations of monomials of ψ fields using the analogue of (4.14): if $\tilde{\psi}(P_{v_i}) = \prod_{f \in P_{v_i}} \psi_{\mathbf{x}(f), \omega(f)}^{\alpha(f)(h_v)}$, the following identity holds:

$$\mathcal{E}_{h_v}^T(\tilde{\psi}(P_{v_1}), \dots, \tilde{\psi}(P_{v_s})) = \left(\frac{1}{Z_{h_v-1}}\right)^n \sum_{T_v} \alpha_{T_v} \prod_{\ell \in T_v} g^{(h_v)}(f_\ell^1, f_\ell^2) \int dP_{T_v}(\mathbf{t}) \text{Pf } G^{T_v}(\mathbf{t}) \quad (5.37)$$

where $g^{(h)}(f, f') = g_{\underline{a}(f), \underline{a}(f')}(\mathbf{x}(f) - \mathbf{x}(f'))$ and the other symbols in (5.37) have the same meaning as those in (4.14).

Using iteratively (5.37) we can express the kernels of $\mathcal{V}^{(h)}$ as sums of products of propagators of the fields (the ones associated to the anchored trees T_v) and Pfaffians of matrices G^{T_v} .

5.7. If the \mathcal{R} operator were not applied to the vertices $v \in \tau$ then the result of the iteration would lead to the following relation:

$$\mathcal{V}_h^*(\tau, \sqrt{Z_h}\psi^{(\leq h)}) = \sqrt{Z_h}^{|P_{v_0}|} \sum_{\mathbf{P} \in \mathcal{P}_\tau} \sum_{T \in \mathbf{T}} \int d\mathbf{x}_{v_0} W_{\tau, \mathbf{P}, \mathbf{T}}^*(\mathbf{x}_{v_0}) \left\{ \prod_{f \in P_{v_0}} \psi_{\mathbf{x}(f), \omega(f)}^{\alpha(f)(\leq h)} \right\}, \quad (5.38)$$

where \mathbf{x}_{v_0} is the set of integration variables associated to τ and $T = \bigcup_v T_v$; $W_{\tau, \mathbf{P}, \mathbf{T}}^*$ is given by

$$\begin{aligned} W_{\tau, \mathbf{P}, \mathbf{T}}^*(\mathbf{x}_{v_0}) &= \left[\prod_{v \text{ not e.p.}} \left(\frac{Z_{h_v}}{Z_{h_v-1}} \right)^{\frac{|P_v|}{2}} \right] \left[\prod_{i=1}^n K_{v_i^*}^{h_i}(\mathbf{x}_{v_i^*}) \right] \left\{ \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \cdot \right. \\ &\quad \left. \cdot \text{Pf } G^{h_v, T_v}(\mathbf{t}_v) \left[\prod_{l \in T_v} g^{(h_v)}(f_l^1, f_l^2) \right] \right\}, \end{aligned} \quad (5.39)$$

where: *e.p.* is an abbreviation of “end points”; v_1^*, \dots, v_n^* are the endpoints of τ , $h_i \equiv h_{v_i^*}$ and $K_{v_i^*}^{h_i}(\mathbf{x}_{v_i^*})$ are the corresponding kernels (equal to $\lambda_{h_v-1}\delta(\mathbf{x}_v)$ or $\nu_{h_v-1}\delta(\mathbf{x}_v)$ if v is an endpoint of type λ or ν on scale $h_v \leq 1$; or equal to one of the kernels of $\mathcal{V}^{(1)}$ if $h_v = 2$).

Bounding (5.39) using (5.24) and the Gram–Hadamard inequality, see Appendix A3, we would find:

$$\int d\mathbf{x}_{v_0} |W_{\tau, \mathbf{P}, \mathbf{T}}^*(\mathbf{x}_{v_0})| \leq C^n M^2 |\lambda|^n \gamma^{-h(-2+|P_{v_0}|/2)} \prod_{v \text{ not e.p.}} \left\{ \frac{1}{s_v!} \left(\frac{Z_{h_v}}{Z_{h_v-1}} \right)^{\frac{|P_v|}{2}} \gamma^{-[-2+\frac{|P_v|}{2}]} \right\}. \quad (5.40)$$

We call $D_v = -2 + \frac{|P_v|}{2}$ the *dimension* of v , depending on the number of the external fields of v . If $D_v < 0$ for any v one can sum over τ, \mathbf{P}, T obtaining convergence for λ small enough; however $D_v \leq 0$ when there are two or four external lines. We will take now into account the effect of the \mathcal{R} operator and we will see how the bound (5.40) is improved.

5.8. The effect of application of \mathcal{P}_j and \mathcal{S}_j is to replace a kernel $W_{2n, \underline{\sigma}, \underline{j}, \underline{\alpha}, \underline{\omega}}^{(h)}$ with $\mathcal{P}_j W_{2n, \underline{\sigma}, \underline{j}, \underline{\alpha}, \underline{\omega}}^{(h)}$ and $\mathcal{S}_j W_{2n, \underline{\sigma}, \underline{j}, \underline{\alpha}, \underline{\omega}}^{(h)}$. If inductively, starting from the end-points, we write the kernels $W_{2n, \underline{\sigma}, \underline{j}, \underline{\alpha}, \underline{\omega}}^{(h)}$ in a form similar to (5.39), we easily realize that, eventually, \mathcal{P}_j or \mathcal{S}_j will act on some propagator of an anchored tree or on

some Pfaffian $\text{Pf } G^{T_v}$, for some v . It is easy to realize that \mathcal{P}_j and \mathcal{S}_j , when applied to Pfaffians, do not break the Pfaffian structure. In fact the effect of \mathcal{P}_j on the Pfaffian of an antisymmetric matrix G with elements $G_{f,f'}$, $f, f' \in J$, $|J| = 2k$, is the following (the proof is trivial):

$$\mathcal{P}_0 \text{Pf } G = \text{Pf } G^0, \quad \mathcal{P}_1 \text{Pf } G = \frac{1}{2} \sum_{f_1, f_2 \in J} \mathcal{P}_1 G_{f_1, f_2} (-1)^\pi \text{Pf } G_1^0, \quad (5.41)$$

where G^0 is the matrix with elements $\mathcal{P}_0 G_{f, f'}$, $f, f' \in J$; G_1^0 is the matrix with elements $\mathcal{P}_0 G_{f, f'}$, $f, f' \in J_1 \stackrel{\text{def}}{=} J \setminus \{f_1 \cup f_2\}$ and $(-1)^\pi$ is the sign of the permutation leading from the ordering J of the labels f in the l.h.s. to the ordering f_1, f_2, J_1 in the r.h.s. The effect of \mathcal{S}_j is the following, see Appendix A5 for a proof:

$$\mathcal{S}_1 \text{Pf } G = \frac{1}{2 \cdot k!} \sum_{f_1, f_2 \in J} \mathcal{S}_1 G_{f_1, f_2} \sum_{J_1 \cup J_2 = J \setminus \cup_i f_i}^* (-1)^\pi k_1! k_2! \text{Pf } G_1^0 \text{Pf } G_2, \quad (5.42)$$

where: the $*$ on the sum means that $J_1 \cap J_2 = \emptyset$; $|J_i| = 2k_i$, $i = 1, 2$; $(-1)^\pi$ is the sign of the permutation leading from the ordering J of the fields labels on the l.h.s. to the ordering f_1, f_2, J_1, J_2 on the r.h.s.; G_1^0 is the matrix with elements $\mathcal{P}_0 G_{f, f'}$, $f, f' \in J_1$; G_2 is the matrix with elements $G_{f, f'}$, $f, f' \in J_2$. The effect of \mathcal{S}_2 on $\text{Pf } G^T$ is given by a formula similar to (5.42). Note that the number of terms in the sums appearing in (5.41), (5.42) (and in the analogous equation for $\mathcal{S}_2 \text{Pf } G^T$), is bounded by c^k for some constant c .

5.9. It is possible to show that the \mathcal{R}_j operators produce derivatives applied to the propagators of the anchored trees and on the elements of G^{T_v} ; and a product of “zeros” of the form $d_j^b(\mathbf{x}(f_\ell^1) - \mathbf{x}(f_\ell^2))$, $j = 0, 1$, $b = 0, 1, 2$, associated to the lines $\ell \in T_v$. This is a well known result, and a very detailed discussion can be found in §3 of [BM]. By such analysis, and using (5.41), (5.42), we get the following expression for $\mathcal{R}\mathcal{V}^{(h)}(\tau, \sqrt{Z_h} \psi^{(\leq h)})$:

$$\begin{aligned} \mathcal{R}\mathcal{V}^{(h)}(\tau, \sqrt{Z_h} \psi^{(\leq h)}) &= \\ &= \sqrt{Z_h}^{|P_{v_0}|} \sum_{\mathbf{P} \in \mathcal{P}_\tau} \sum_{T \in \mathbf{T}} \sum_{\beta \in B_T} \int d\mathbf{x}_{v_0} W_{\tau, \mathbf{P}, \mathbf{T}, \beta}(\mathbf{x}_{v_0}) \left\{ \prod_{f \in P_{v_0}} \hat{\partial}_{j_\beta(f)}^{q_\beta(f)} \psi_{\mathbf{x}_\beta(f), \omega(f)}^{\alpha(f)(\leq h)} \right\}, \end{aligned} \quad (5.43)$$

where: B_T is a set of indices which allows to distinguish the different terms produced by the non trivial \mathcal{R} operations; $\mathbf{x}_\beta(f)$ is a coordinate obtained by interpolating two points in \mathbf{x}_{v_0} , in a suitable way depending on β ; $q_\beta(f)$ is a nonnegative integer ≤ 2 ; $j_\beta(f) = 0, 1$ and $\hat{\partial}_j^q$ is a suitable differential operator, dimensionally equivalent to ∂_j^q (see [BM] for a precise definition); $W_{\tau, \mathbf{P}, \mathbf{T}, \beta}$ is given by:

$$\begin{aligned} W_{\tau, \mathbf{P}, \mathbf{T}, \beta}(\mathbf{x}_{v_0}) &= \left[\prod_{v \text{ not e.p.}} \left(\frac{Z_{h_v}}{Z_{h_v-1}} \right)^{\frac{|P_v|-1}{2}} \right] \left[\prod_{i=1}^n d_{j_\beta(v_i^*)}^{b_\beta(v_i^*)}(\mathbf{x}_\beta^i, \mathbf{y}_\beta^i) \mathcal{P}_{I_\beta(v)}^{C_\beta(v_i^*)} \mathcal{S}_{i_\beta(v_i^*)}^{c_\beta(v_i^*)} K_{v_i^*}^{h_i}(\mathbf{x}_{v_i^*}) \right] \cdot \\ &\cdot \left\{ \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \mathcal{P}_{I_\beta(v)}^{C_\beta(v)} \mathcal{S}_{i_\beta(v)}^{c_\beta(v)} \text{Pf } G_\beta^{h_v, T_v}(\mathbf{t}_v) \cdot \right. \\ &\cdot \left. \left[\prod_{l \in T_v} \hat{\partial}_{j_\beta(f_l^1)}^{q_\beta(f_l^1)} \hat{\partial}_{j_\beta(f_l^2)}^{q_\beta(f_l^2)} [d_{j_\beta(l)}^{b_\beta(l)}(\mathbf{x}_l, \mathbf{y}_l) \mathcal{P}_{I_\beta(l)}^{C_\beta(l)} \mathcal{S}_{i_\beta(l)}^{c_\beta(l)} g^{(h_v)}(f_l^1, f_l^2)] \right] \right\}, \end{aligned} \quad (5.44)$$

where: v_1^*, \dots, v_n^* are the endpoints of τ ; $b_\beta(v)$, $b_\beta(l)$, $q_\beta(f_l^1)$ and $q_\beta(f_l^2)$ are nonnegative integers ≤ 2 ; $j_\beta(v)$, $j_\beta(f_l^1)$, $j_\beta(f_l^2)$ and $j_\beta(l)$ can be 0 or 1; $i_\beta(v)$ and $i_\beta(l)$ can be 1 or 2; $I_\beta(v)$ and $I_\beta(l)$ can be 0 or 1; $C_\beta(v)$, $c_\beta(v)$, $C_\beta(l)$ and $c_\beta(l)$ can be 0, 1 and $\max\{C_\beta(v) + c_\beta(v), C_\beta(l) + c_\beta(l)\} \leq 1$; $G_\beta^{h_v, T_v}(\mathbf{t}_v)$ is obtained from $G^{h_v, T_v}(\mathbf{t}_v)$ by substituting the element $t_{i(f), i(f')} g^{(h_v)}(f, f')$ with $t_{i(f), i(f')} \hat{\partial}_{j_\beta(f)}^{q_\beta(f)} \hat{\partial}_{j_\beta(f')}^{q_\beta(f')} g^{(h_v)}(f, f')$.

It would be very difficult to give a precise description of the various contributions of the sum over B_T , but fortunately we only need to know some very general properties, which easily follows from the construction in §5.1–§5.3.

1) There is a constant C such that, $\forall T \in \mathbf{T}_\tau$, $|B_T| \leq C^n$; for any $\beta \in B_T$, the following inequality is satisfied

$$\left[\prod_{f \in \cup_v P_v} \gamma^{h(f)q_\beta(f)} \right] \left[\prod_{l \in T} \gamma^{-h(l)b_\beta(l)} \right] \leq \prod_{v \text{ not e.p.}} \gamma^{-z(P_v)}, \quad (5.45)$$

where: $h(f) = h_{v_0} - 1$ if $f \in P_{v_0}$, otherwise it is the scale of the vertex where the field with label f is contracted; $h(l) = h_v$, if $l \in T_v$ and

$$z(P_v) = \begin{cases} 1 & \text{if } |P_v| = 4 \text{ and } \rho_v = p, \\ 2 & \text{if } |P_v| = 2 \text{ and } \rho_v = p, \\ 1 & \text{if } |P_v| = 2, \rho_v = s \text{ and } \sum_{f \in P_v} \omega(f) \neq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (5.46)$$

2) If we define

$$\prod_{v \in \tau} \left[\left(\frac{|\sigma_{h_v}| + |\mu_{h_v}|}{\gamma^{h_v}} \right)^{c_\beta(v)i_\beta(v)} \prod_{\ell \in T_v} \left(\frac{|\sigma_{h_v}| + |\mu_{h_v}|}{\gamma^{h_v}} \right)^{c_\beta(\ell)i_\beta(\ell)} \right] \stackrel{\text{def}}{=} \prod_{v \in V_\beta} \left(\frac{|\sigma_{h_v}| + |\mu_{h_v}|}{\gamma^{h_v}} \right)^{i(v,\beta)}. \quad (5.47)$$

the indeces $i(v, \beta)$ satisfy, for any B_T , the following property:

$$\sum_{w \geq v} i(w, \beta) \geq z'(P_v), \quad (5.48)$$

where

$$z'(P_v) = \begin{cases} 1 & \text{if } |P_v| = 4 \text{ and } \rho_v = s, \\ 2 & \text{if } |P_v| = 2 \text{ and } \rho_v = s \text{ and } \sum_{f \in P_v} \omega(f) = 0, \\ 1 & \text{if } |P_v| = 2, \rho_v = s \text{ and } \sum_{f \in P_v} \omega(f) \neq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (5.49)$$

5.10. We can bound any $|\mathcal{P}_{I_\beta(v)}^{C_\beta(v)} \mathcal{S}_{i_\beta(v)}^{c_\beta(v)} \text{Pf } G_\beta^{h_v, T_v}|$ in (5.44), with $C_\beta(v) + c_\beta(v) = 0, 1$, by using (5.41), (5.42) and Gram inequality, as illustrated in previous Chapter for the case of the integration of the χ fields. Using that the elements of G are all propagators on scale h_v , dimensionally bounded as in Lemma 5.3, we find:

$$\begin{aligned} |\mathcal{P}_{I_\beta(v)}^{C_\beta(v)} \mathcal{S}_{i_\beta(v)}^{c_\beta(v)} \text{Pf } G_\beta^{h_v, T_v}| &\leq C \sum_{i=1}^{s_v} |P_{v_i}| - |P_v| - 2(s_v - 1). \\ &\cdot \gamma^{\frac{h_v}{2} (\sum_{i=1}^{s_v} |P_{v_i}| - |P_v| - 2(s_v - 1))} \left[\prod_{f \in J_v} \gamma^{h_v q_\beta(f)} \right] \left(\frac{|\sigma_{h_v}| + |\mu_{h_v}|}{\gamma^{h_v}} \right)^{c_\beta(v)i_\beta(v) + C_\beta(v)I_\beta(v)}, \end{aligned} \quad (5.50)$$

where $J_v = \cup_{i=1}^{s_v} P_{v_i} \setminus Q_{v_i}$. We will bound the factors $\left(\frac{|\sigma_{h_v}| + |\mu_{h_v}|}{\gamma^{h_v}} \right)^{C_\beta(v)I_\beta(v)}$ using (5.23) times a constant.

If we call

$$\begin{aligned} J_{\tau, \mathbf{P}, T, \beta} &= \int d\mathbf{x}_{v_0} \left| \left[\prod_{i=1}^n d_{j_\beta(v_i^*)}^{b_\beta(v_i^*)}(\mathbf{x}_\beta^i, \mathbf{y}_\beta^i) \mathcal{P}_{I_\beta(v_i^*)}^{C_\beta(v_i^*)} \mathcal{S}_{i_\beta(v_i^*)}^{c_\beta(v_i^*)} K_{v_i^*}^{h_i}(\mathbf{x}_{v_i^*}) \right] \cdot \right. \\ &\quad \cdot \left. \left\{ \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \left[\prod_{l \in T_v} \hat{\partial}_{j_\beta(f_l^1)}^{q_\beta(f_l^1)} \hat{\partial}_{j_\beta(f_l^2)}^{q_\beta(f_l^2)} [d_{j_\beta(l)}^{b_\beta(l)}(\mathbf{x}_l, \mathbf{y}_l) \mathcal{P}_{I_\beta(l)}^{C_\beta(l)} \mathcal{S}_{i_\beta(l)}^{c_\beta(l)} g^{(h_v)}(f_l^1, f_l^2)] \right] \right\} \right|, \end{aligned} \quad (5.51)$$

we have, under the hypothesis (5.29),

$$\begin{aligned}
 J_{\tau, \mathbf{P}, T, \alpha} &\leq C^n M^2 |\lambda|^n \left[\prod_{i=1}^n \left(\frac{|\sigma_{h_i^*}| + |\mu_{h_i^*}|}{\gamma^{h_i^*}} \right)^{c_\beta(v_i^*) i_\beta(v_i^*)} \right] \\
 &\cdot \left\{ \prod_{v \text{ not e.p.}} \frac{1}{s_v!} C^{2(s_v-1)} \gamma^{h_v n_\nu(v)} \gamma^{-h_v \sum_{l \in T_v} b_\beta(l)} \gamma^{-h_v \sum_{i=1}^n b_\beta(v_i^*)} \gamma^{-h_v(s_v-1)} \right. \\
 &\cdot \gamma^{h_v \sum_{l \in T_v} [q_\beta(f_l^1) + q_\beta(f_l^2)]} \left. \right\} \left[\prod_{\ell \in T} \left(\frac{|\sigma_{h_\ell}| + |\mu_{h_\ell}|}{\gamma^{h_\ell}} \right)^{c_\beta(\ell) i_\beta(\ell)} \right], \tag{5.52}
 \end{aligned}$$

where $n_\nu(v)$ is the number of vertices of type ν with scale $h_v + 1$.

Now, substituting (5.50), (5.52) into (5.44), using (5.45), we find that:

$$\begin{aligned}
 \int d\mathbf{x}_{v_0} |W_{\tau, \mathbf{P}, T, \beta}(\mathbf{x}_{v_0})| &\leq C^n M^2 |\lambda|^n \gamma^{-h D_k(|P_{v_0}|)} \prod_{v \in V_\beta} \left(\frac{|\sigma_{h_v}| + |\mu_{h_v}|}{\gamma^{h_v}} \right)^{i(v, \beta)} \\
 &\cdot \prod_{v \text{ not e.p.}} \left\{ \frac{1}{s_v!} C^{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|} \left(\frac{Z_{h_v}}{Z_{h_v-1}} \right)^{\frac{|P_v|}{2}} \gamma^{-[-2 + \frac{|P_v|}{2} + z(P_v)]} \right\}, \tag{5.53}
 \end{aligned}$$

where, if $k = \sum_{f \in P_{v_0}} q_\beta(f)$, $D_k(p) = -2 + p + k$ and we have used (5.47). Note that, given $v \in \tau$ and $\tau \in \mathcal{T}_{h,n}$ and using (5.23) together with the first two of (5.22),

$$\begin{aligned}
 \frac{|\sigma_{h_v}|}{\gamma^{h_v}} &= \frac{|\sigma_h|}{\gamma^h} \frac{|\sigma_{h_v}|}{|\sigma_h|} \gamma^{h-h_v} \leq \frac{|\sigma_h|}{\gamma^h} \gamma^{(h-h_v)(1-c|\lambda|)} \leq C_1 \gamma^{(h-h_v)(1-c|\lambda|)} \\
 \frac{|\mu_{h_v}|}{\gamma^{h_v}} &= \frac{|\mu_h|}{\gamma^h} \frac{|\mu_{h_v}|}{|\mu_h|} \gamma^{h-h_v} \leq \frac{|\mu_h|}{\gamma^h} \gamma^{(h-h_v)(1-c|\lambda|)} \leq C_1 \gamma^{(h-h_v)(1-c|\lambda|)} \tag{5.54}
 \end{aligned}$$

Moreover the indices $i(v, \beta)$ satisfy, for any B_T , (5.49) so that, using (5.54) and (5.48), we find

$$\prod_{v \in V_\beta} \left(\frac{|\sigma_{h_v}| + |\mu_{h_v}|}{\gamma^{h_v}} \right)^{i(v, \beta)} \leq C_1^n \prod_{v \text{ not e.p.}} \gamma^{-(1-c|\lambda|)z'(P_v)}. \tag{5.55}$$

Substituting (5.54) into (5.53) and using (5.48), we find:

$$\begin{aligned}
 \int d\mathbf{x}_{v_0} |W_{\tau, \mathbf{P}, T, \beta}(\mathbf{x}_{v_0})| &\leq C^n M^2 |\lambda|^n \gamma^{-h D_k(|P_{v_0}|)} \\
 &\cdot \prod_{v \text{ not e.p.}} \left\{ \frac{1}{s_v!} C^{\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|} \left(\frac{Z_{h_v}}{Z_{h_v-1}} \right)^{\frac{|P_v|}{2}} \gamma^{-[-2 + \frac{|P_v|}{2} + z(P_v) + (1-c|\lambda|)z'(P_v)]} \right\}. \tag{5.56}
 \end{aligned}$$

and it holds:

$$-2 + \frac{|P_v|}{2} + z(P_v) + (1-c|\lambda|)z'(P_v) \geq \frac{|P_v|}{6}. \tag{5.57}$$

Then (5.30) in Theorem 5.1 follows from the previous bounds and the remark that

$$\sum_{\tau \in \mathcal{T}_{h,n}} \sum_{\mathbf{P} \in \mathcal{P}_\tau} \sum_{T \in \mathbf{T}} \sum_{\beta \in B_T} \prod_v \frac{1}{s_v!} \gamma^{-\frac{|P_v|}{6}} \leq c^n, \tag{5.58}$$

for some constant c , see [BM][GM] or [G] for further details.

The bound on \tilde{E}_h , t_h , (5.31) and (5.32) follow from a similar analysis. The remarks following (5.31) and (5.32) follow from noticing that in the expansion for $\mathcal{LV}^{(h)}$ appear only propagators of type $\mathcal{P}_0 g_{\underline{a}, \underline{a}'}^{(h_v)}$ or $\mathcal{P}_1 g_{\underline{a}, \underline{a}'}^{(h_v)}$ (in order to bound these propagators we do not need (5.23), see the last statement in Lemma 5.3). Furthermore, by construction l_h, n_h and z_h are independent of σ_k, μ_k , so that, in order to prove (5.32) we do not even need the first two inequalities in (5.22). ■

5.11. The sum over all the trees with root scale h and with at least a v with $h_v = k$ is $O(|\lambda| \gamma^{\frac{1}{2}(h-k)})$; this follows from the fact that the bound (5.58) holds, for some $c = O(1)$, even if $\gamma^{-|P_v|/6}$ is replaced by $\gamma^{-\kappa|P_v|}$, for any constant $\kappa > 0$ independent of λ ; and that D_v , instead of using (5.57), can also be bounded as $D_v \geq 1/2 + |P_v|/12$. This property is called *short memory property*.

6. The flow of the running coupling constants.

The convergence of the expansion for the effective potential is proved by Theorem 5.1 under the hypothesis that the running coupling constants are small, see (5.29), and that the bounds (5.21), (5.22) and (5.23) are satisfied. We now want to show that, choosing λ small enough and ν as a suitable function of λ , such hypothesis are indeed verified. In the present Chapter, we will prove these hypotheses under the assumption that the Luttinger model Beta function is vanishing; we will do more, and we will find an explicit solution for the flow equation of Z_h, σ_h, μ_h , satisfying in particular the bounds (5.21), (5.22) and satisfying (5.23) for any scale $\bar{h} \geq h_1^*$, where h_1^* is a scale we will explicitly choose in the present Chapter (it is the scale dividing the anomalous regime from the non anomalous one). The proof of the vanishing of the Beta function will be done in Appendix A6, following the recent work [BM1]. The proof of Appendix A6 will be based on the implementation in our constructive formalism of some non perturbative identities between Schwinger functions, that is of two different approximate Ward identities for the two and four legs Schwinger functions respectively, of the Dyson equation, and of some correction identities, expressing the corrections to the formal Ward identities in terms of two or four legs Schwinger functions. It worths to stress that these non perturbative identities are derived by making use of (chiral) gauge invariance, that is *not* satisfied by the Ashkin–Teller model. However, since there is a model near to AT in a Renormalization Group sense (we shall call it the *reference model*) satisfying these symmetries, the cancellations appearing in the perturbation theory of the reference model also imply cancellations for AT itself. We can say that some *hidden* symmetries of Ashkin–Teller allow us to control the flow of its running coupling constants. Note that here the word “hidden” has a different (and much deeper) meaning than in the introduction of Chapter 4.

6.1. The flow equations

We will first solve the flow equations for the renormalization constants (following from (5.14) and preceding line):

$$\frac{Z_{h-1}}{Z_h} = 1 + z_h \quad , \quad \frac{\sigma_{h-1}}{\sigma_h} = 1 + \frac{s_h/\sigma_h - z_h}{1 + z_h} \quad , \quad \frac{\mu_{h-1}}{\mu_h} = 1 + \frac{m_h/\mu_h - z_h}{1 + z_h} \quad , \quad (6.1)$$

together with those for the running coupling constants (5.28):

$$\begin{aligned} \lambda_{h-1} &= \lambda_h + \beta_\lambda^h(\lambda_h, \nu_h; \dots; \lambda_1, \nu_1) \\ \nu_{h-1} &= \gamma \nu_h + \beta_\nu^h(\lambda_h, \nu_h; \dots; \lambda_1, \nu_1) . \end{aligned} \quad (6.2)$$

The functions $\beta_\lambda^h, \beta_\nu^h$ are called the λ and ν components of the Beta function, see the comment after (5.27), and, by construction, are *independent* of σ_k, μ_k , so that their convergence follow just from (5.29) and the last of (5.22), *i.e.* without assuming (5.23), see Theorem 5.1. While for a general kernel we will apply Theorem 5.1 just up to a finite scale h_1^* (in order to insure the validity of (5.23) with $\bar{h} = h_1^*$), we will inductively study the flow generated by (6.2) up to scale $-\infty$, and we shall prove that it is bounded for all scales. The main result on the flows of λ_h and ν_h , proven in next section, is the following.

THEOREM 6.1. *If λ is small enough, there exists an analytic function $\nu^*(\lambda)$ independent of t, u such that the running coupling constants $\{\lambda_h, \nu_h\}_{h \leq 1}$ with $\nu_1 = \nu^*(\lambda)$ verify $|\nu_h| \leq c|\lambda|\gamma^{(\vartheta/2)h}$ and $|\lambda_h| \leq c|\lambda|$. Moreover the kernels z_h, s_h and m_h satisfy (5.21) and the solutions of the flow equations (6.1) satisfy (5.22).*

6.2. Proof of Theorem 6.1.

We consider the space \mathfrak{M}_ϑ of sequences $\underline{\nu} = \{\nu_h\}_{h \leq 1}$ such that $|\nu_h| \leq c|\lambda|\gamma^{(\vartheta/2)h}$; we shall think \mathfrak{M}_ϑ as a Banach space with norm $\|\cdot\|_\vartheta$, where $\|\underline{\nu}\|_\vartheta \stackrel{\text{def}}{=} \sup_{k \leq 1} |\nu_k| \gamma^{-(\vartheta/2)k}$. We will proceed as follows: we first

show that, for any sequence $\underline{\nu} \in \mathfrak{M}_\vartheta$, the flow equation for ν_h , the hypothesis (5.21), (5.22) and the property $|\lambda_h(\underline{\nu})| \leq c|\lambda|$ are verified, uniformly in $\underline{\nu}$. Then we fix $\underline{\nu} \in \mathfrak{M}_\vartheta$ via an exponentially convergent iterative procedure, in such a way that the flow equation for ν_h is satisfied.

Given $\underline{\nu} \in \mathfrak{M}_\vartheta$, let us suppose inductively that (5.21), (5.22) and that, for $k > \bar{h} + 1$,

$$|\lambda_{k-1}(\underline{\nu}) - \lambda_k(\underline{\nu})| \leq c_0 |\lambda|^2 \gamma^{(\vartheta/2)^k}, \quad (6.3)$$

for some $c_0 > 0$. Note that (6.3) is certainly true for $h = 1$ (in that case the r.h.s. of (6.3) is just the bound on β_λ^1). Note also that (6.3) implies that $|\lambda_k| \leq c|\lambda|$, for any $k > \bar{h}$.

Using (5.31), the second of (5.32) and (6.1) we find that (5.21), (5.22) are true with \bar{h} replaced by $\bar{h} - 1$.

We now consider the equation $\lambda_{h-1} = \lambda_h + \beta_\lambda^h(\lambda_h, \nu_h; \dots; \lambda_1, \nu_1)$, $h > \bar{h}$. The function β_λ^h can be expressed as a convergent sum over tree diagrams, as described in §5.5; note that it depends on $(\lambda_h, \nu_h; \dots; \lambda_1, \nu_1)$ directly through the end-points of the trees and indirectly through the factors Z_h/Z_{h-1} .

We can write $\mathcal{P}_0 g_{(+,\omega),(-,\omega)}^{(h)}(\mathbf{x} - \mathbf{y}) = g_{L,\omega}^{(h)}(\mathbf{x} - \mathbf{y}) + r_\omega^{(h)}(\mathbf{x} - \mathbf{y})$, where

$$g_{L,\omega}^{(h)}(\mathbf{x} - \mathbf{y}) \stackrel{\text{def}}{=} \frac{4}{M^2} \sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \tilde{f}_h(\mathbf{k}) \frac{1}{ik + \omega k_0} \quad (6.4)$$

and $r_\omega^{(h)}$ is the rest, satisfying the same bound as $g_{(+,\omega),(-,\omega)}^{(h)}$, times a factor γ^h . This decomposition induces the following decomposition for β_λ^h :

$$\begin{aligned} \beta_\lambda^h(\lambda_h, \nu_h; \dots; \lambda_1, \nu_1) &= \\ &= \beta_{\lambda,L}^h(\lambda_h, \dots, \lambda_h) + \sum_{k=h+1}^1 D_\lambda^{h,k} + r_\lambda^h(\lambda_h, \dots, \lambda_1) + \sum_{k \geq h} \nu_k \tilde{\beta}_\lambda^{h,k}(\lambda_k, \nu_k; \dots; \lambda_1, \nu_1), \end{aligned} \quad (6.5)$$

with

$$\begin{aligned} |\beta_{\lambda,L}^h| &\leq c|\lambda|^2 \gamma^{\vartheta h}, & |D_\lambda^{h,k}| &\leq c|\lambda| \gamma^{\vartheta(h-k)} |\lambda_k - \lambda_h|, \\ |r_\lambda^h| &\leq c|\lambda|^2 \gamma^{(\vartheta/2)h}, & |\tilde{\beta}_\lambda^{h,k}| &\leq c|\lambda| \gamma^{\vartheta(h-k)}. \end{aligned} \quad (6.6)$$

The first two terms in (6.5) $\beta_{\lambda,L}^h$ collect the contributions obtained by posing $r_\omega^{(k)} = 0$, $k \geq h$ and substituting the discrete δ function defined after (5.10) with $M^2 \delta_{\mathbf{k},0}$. The first of (6.6) is called the *vanishing of the Luttinger model Beta function* property, and it is a crucial and non trivial property of interacting fermionic systems in $d = 1$. It will be proved in Appendix A6.

Using the decomposition (6.5) and the bounds (6.6) we prove the following bounds for $\lambda_{\bar{h}}(\underline{\nu})$, $\underline{\nu} \in \mathfrak{M}_\vartheta$:

$$|\lambda_{\bar{h}}(\underline{\nu}) - \lambda_1(\underline{\nu})| \leq c_0 |\lambda|^2, \quad |\lambda_{\bar{h}}(\underline{\nu}) - \lambda_{\bar{h}+1}(\underline{\nu})| \leq c_0 |\lambda|^2 \gamma^{(\vartheta/2)^{\bar{h}}}, \quad (6.7)$$

for some $c_0 > 0$. Moreover, given $\underline{\nu}, \underline{\nu}' \in \mathfrak{M}_\vartheta$, we show that:

$$|\lambda_{\bar{h}}(\underline{\nu}) - \lambda_{\bar{h}}(\underline{\nu}')| \leq c|\lambda| \|\underline{\nu} - \underline{\nu}'\|_0, \quad (6.8)$$

where $\|\underline{\nu} - \underline{\nu}'\|_0 \stackrel{\text{def}}{=} \sup_{h \leq 1} |\nu_h - \nu'_h|$.

PROOF OF (6.7). We decompose $\lambda_{\bar{h}} - \lambda_{\bar{h}+1} = \beta_\lambda^{\bar{h}+1}$ as in (6.5). Using the bounds (6.6) and the inductive hypothesis (6.3), we find:

$$\begin{aligned} |\lambda_{\bar{h}}(\underline{\nu}) - \lambda_{\bar{h}+1}(\underline{\nu})| &\leq c|\lambda|^2 \gamma^{\vartheta(\bar{h}+1)} + \sum_{k \geq \bar{h}+2} c|\lambda| \gamma^{\vartheta(\bar{h}+1-k)} \sum_{k'=\bar{h}+2}^k c_0 |\lambda|^2 \gamma^{(\vartheta/2)^{k'}} + \\ &\quad + c|\lambda|^2 \gamma^{(\vartheta/2)(\bar{h}+1)} + \sum_{k \geq \bar{h}+1} c^2 |\lambda|^2 \gamma^{(\vartheta/2)^k} \gamma^{\vartheta(\bar{h}+1-k)}, \end{aligned} \quad (6.9)$$

which, for c_0 big enough, immediately implies the second of (6.7) with $h \rightarrow h-1$; from this bound and the hypothesis (6.3) follows the first of (6.7). ■

PROOF OF (6.8). If we take two sequences $\underline{\nu}, \underline{\nu}' \in \mathfrak{M}_\vartheta$, we easily find that the beta function for $\lambda_{\bar{h}}(\underline{\nu}) - \lambda_{\bar{h}}(\underline{\nu}')$ can be represented by a tree expansion similar to the one for β_λ^h , with the property that the trees giving a non vanishing contribution have necessarily one end-point on scale $k \geq h$ associated to a coupling constant $\lambda_k(\underline{\nu}) - \lambda_k(\underline{\nu}')$ or $\nu_k - \nu'_k$. Then we find:

$$\lambda_{\bar{h}}(\underline{\nu}) - \lambda_{\bar{h}}(\underline{\nu}') = \lambda_1(\underline{\nu}) - \lambda_1(\underline{\nu}') + \sum_{\bar{h}+1 \leq k \leq 1} [\beta_\lambda^k(\lambda_k(\underline{\nu}), \nu_k; \dots; \lambda_1, \nu_1) - \beta_\lambda^k(\lambda_k(\underline{\nu}'), \nu'_k; \dots; \lambda_1, \nu'_1)] . \quad (6.10)$$

Note that $|\lambda_1(\underline{\nu}) - \lambda_1(\underline{\nu}')| \leq c_0 |\lambda| |\nu_1 - \nu'_1|$, because $\lambda_1 = \lambda/Z_1^2 + O(\lambda^2/Z_1^4)$ and $Z_1 = \sqrt{2} - 1 + \nu/2$. If we inductively suppose that, for any $k > \bar{h}$, $|\lambda_k(\underline{\nu}) - \lambda_k(\underline{\nu}')| \leq 2c_0 |\lambda| \|\underline{\nu} - \underline{\nu}'\|_0$, we find, by using the decomposition (6.5):

$$|\lambda_{\bar{h}}(\underline{\nu}) - \lambda_{\bar{h}}(\underline{\nu}')| \leq c_0 |\lambda| |\nu_1 - \nu'_1| + c |\lambda| \sum_{k \geq \bar{h}+1} \gamma^{(\vartheta/2)k} \sum_{k' \geq k} \gamma^{\vartheta(k-k')} \left[2c_0 |\lambda| \|\underline{\nu} - \underline{\nu}'\|_0 + |\nu_k - \nu'_k| \right] . \quad (6.11)$$

Choosing c_0 big enough, (6.8) follows. ■

We are now left with fixing the sequence $\underline{\nu}$ in such a way that the flow equation for ν is satisfied. Since we want to fix $\underline{\nu}$ in such a way that $\nu_{-\infty} = 0$, we must have:

$$\nu_1 = - \sum_{k=-\infty}^1 \gamma^{k-2} \beta_\nu^k(\lambda_k, \nu_k; \dots; \lambda_1, \nu_1) . \quad (6.12)$$

If we manage to fix ν_1 as in (6.12), we also get:

$$\nu_h = - \sum_{k \leq h} \gamma^{k-h-1} \beta_\nu^k(\lambda_k, \nu_k; \dots; \lambda_1, \nu_1) . \quad (6.13)$$

We look for a fixed point of the operator $\mathbf{T} : \mathfrak{M}_\vartheta \rightarrow \mathfrak{M}_\vartheta$ defined as:

$$(\mathbf{T}\underline{\nu})_h = - \sum_{k \leq h} \gamma^{k-h-1} \beta_\nu^k(\lambda_k(\underline{\nu}), \nu_k; \dots; \lambda_1, \nu_1) . \quad (6.14)$$

where $\lambda_k(\underline{\nu})$ is the solution of the first line of (6.2), obtained as a function of the *parameter* $\underline{\nu}$, as described above.

If we find a fixed point $\underline{\nu}^*$ of (6.14), the first two lines in (6.2) will be simultaneously solved by $\underline{\lambda}(\underline{\nu}^*)$ and $\underline{\nu}^*$ respectively, and the solution will have the desired smallness properties for λ_h and ν_h .

First note that, if $|\lambda|$ is sufficiently small, then \mathbf{T} leaves \mathfrak{M}_ϑ invariant: in fact, as a consequence of parity cancellations, the ν -component of the Beta function satisfies:

$$\beta_\nu^h(\lambda_h, \nu_h; \dots; \lambda_1, \nu_1) = \beta_{\nu,1}^h(\lambda_h; \dots; \lambda_1) + \sum_k \nu_k \tilde{\beta}_\nu^{h,k}(\lambda_h, \nu_h; \dots; \lambda_1, \nu_1) \quad (6.15)$$

where, if c_1, c_2 are suitable constants

$$|\beta_{\nu,1}^h| \leq c_1 |\lambda| \gamma^{\vartheta h} \quad |\tilde{\beta}_\nu^{h,k}| \leq c_2 |\lambda| \gamma^{\vartheta(h-k)} . \quad (6.16)$$

by using (6.15) and choosing $c = 2c_1$ we obtain

$$|(\mathbf{T}\nu)_h| \leq \sum_{k \leq h} 2c_1 |\lambda| \gamma^{(\vartheta/2)k} \gamma^{k-h} \leq c |\lambda| \gamma^{(\vartheta/2)h}, \quad (6.17)$$

Furthermore, using (6.15) and (6.8), we find that \mathbf{T} is a contraction on \mathfrak{M}_ϑ :

$$\begin{aligned} |(\mathbf{T}\nu)_h - (\mathbf{T}\nu')_h| &\leq \sum_{k \leq h} \gamma^{k-h-1} |\beta_\nu^k(\lambda_k(\underline{\nu}), \nu_k; \dots; \lambda_1, \nu_1) - \beta_{\nu'}^k(\lambda_k(\underline{\nu}'), \nu'_k; \dots; \lambda_1, \nu'_1)| \leq \\ &\leq c \sum_{k \leq h} \gamma^{k-h-1} \left[\gamma^{\vartheta k} \sum_{k'=k}^1 |\lambda_{k'}(\underline{\nu}) - \lambda_{k'}(\underline{\nu}')| + \sum_{k'=k}^1 \gamma^{\vartheta(k-k')} |\lambda| |\nu_{k'} - \nu'_{k'}| \right] \leq \\ &\leq c' \sum_{k \leq h} \gamma^{k-h-1} \left[|k| \gamma^{\vartheta k} |\lambda| \|\underline{\nu} - \underline{\nu}'\|_0 + \sum_{k'=k}^1 \gamma^{\vartheta(k-k')} |\lambda| \gamma^{(\vartheta/2)k'} \|\underline{\nu} - \underline{\nu}'\|_\vartheta \right] \leq \\ &\leq c'' |\lambda| \gamma^{(\vartheta/2)h} \|\underline{\nu} - \underline{\nu}'\|_\vartheta. \end{aligned} \quad (6.18)$$

hence $\|(\mathbf{T}\nu) - (\mathbf{T}\nu')\|_\vartheta \leq c'' |\lambda| \|\underline{\nu} - \underline{\nu}'\|_\vartheta$. Then, a unique fixed point $\underline{\nu}^*$ for \mathbf{T} exists on \mathfrak{M}_ϑ . Proof of Theorem 6.1 is concluded by noticing that \mathbf{T} is analytic (in fact β_ν^h and $\underline{\lambda}$ are analytic in $\underline{\nu}$ in the domain \mathfrak{M}_ϑ). ■

6.3. The flow of the renormalization constants.

Once that ν_1 is conveniently chosen as in Theorem 6.1, one can study in more detail the flows of the renormalization constants. We will now prove the following.

LEMMA 6.1. *If λ is small enough and ν_1 is chosen as in Theorem 6.1, the solution of (6.1) can be written as:*

$$Z_h = \gamma^{\eta_z(h-1)+F_\zeta^h}, \quad \mu_h = \mu_1 \gamma^{\eta_\mu(h-1)+F_\mu^h}, \quad \sigma_h = \sigma_1 \gamma^{\eta_\sigma(h-1)+F_\sigma^h} \quad (6.19)$$

where $\eta_z, \eta_\mu, \eta_\sigma$ and $F_\zeta^h, F_\mu^h, F_\sigma^h$ are $O(\lambda)$ functions, independent of σ_1, μ_1 .

Moreover $\eta_\sigma - \eta_\mu = -b\lambda + O(|\lambda|^2)$, $b > 0$.

PROOF OF LEMMA 6.1 From now on we shall think λ_h and ν_h fixed, with ν_1 conveniently chosen as above ($\nu_1 = \nu_1^*(\lambda)$). Then we have $|\lambda_h| \leq c|\lambda|$ and $|\nu_h| \leq c|\lambda| \gamma^{(\vartheta/2)h}$, for some $c, \vartheta > 0$. Having fixed ν_1 as a convenient function of λ , we can also think λ_h and ν_h as functions of λ .

The flow of Z_h . The flow of Z_h is given by the first of (6.1) with z_h independent of σ_k, μ_k , $k \geq h$. By Theorem 3.1 we have that $|z_h| \leq c|\lambda|^2$, uniformly in h . Again, as for λ_h and ν_h , we can formally study this equation up to $h = -\infty$. We define $\gamma^{-\eta_z} \stackrel{\text{def}}{=} \lim_{h \rightarrow -\infty} 1 + z_h$, so that

$$\log_\gamma Z_h = \sum_{k \geq h+1} \log_\gamma(1 + z_k) = \eta_z(h-1) + \sum_{k \geq h+1} r_\zeta^k, \quad r_\zeta^k \stackrel{\text{def}}{=} \log_\gamma \left(1 + \frac{z_k - z_{-\infty}}{1 + z_{-\infty}} \right). \quad (6.20)$$

Using the fact that $z_{k-1} - z_k$ is necessarily proportional to $\lambda_{k-1} - \lambda_k$ or to $\nu_{k-1} - \nu_k$ and that $\lambda_{k-1} - \lambda_k$ is bounded as in (6.3), we easily find: $|r_\zeta^k| \leq c \sum_{k' \leq k} |z_{k'-1} - z_{k'}| \leq c' |\lambda|^2 \gamma^{(\vartheta/2)k}$. So, if $F_\zeta^h \stackrel{\text{def}}{=} \sum_{k \geq h+1} r_\zeta^k$ and $F_\zeta^1 = 0$, then $F_\zeta^h = O(\lambda)$ and $Z_h = \gamma^{\eta_z(h-1)+F_\zeta^h}$. Clearly, by definition, η_z and F_ζ^h only depend on λ_k, ν_k , $k \leq 1$, so they are independent of t and u .

The flow of μ_h . The flow of μ_h is given by the last of (6.1). One can easily show inductively that $\mu_k(\mathbf{k})/\mu_h$, $k \geq h$, is independent of μ_1 , so that one can think that μ_{h-1}/μ_h is just a function of λ_h, ν_h . Then, again we

can study the flow equation for μ_h up to $h \rightarrow -\infty$. We define $\gamma^{-\eta_\mu} \stackrel{def}{=} \lim_{h \rightarrow -\infty} 1 + (m_h/\mu_h - z_h)/(1 + z_h)$, so that, proceeding as for Z_h , we see that

$$\mu_h = \mu_1 \gamma^{\eta_\mu(h-1) + F_\mu^h}, \quad (6.21)$$

for a suitable $F_\mu^h = O(\lambda)$. Of course η_μ and F_μ^h are independent of t and u .

The flow of σ_h . The flow of σ_h can be studied as the one of μ_h . If we define $\gamma^{-\eta_\sigma} \stackrel{def}{=} \lim_{h \rightarrow -\infty} 1 + (s_h/\sigma_h - z_h)/(1 + z_h)$, we find that

$$\sigma_h = \sigma_1 \gamma^{\eta_\sigma(h-1) + F_\sigma^h}, \quad (6.22)$$

for a suitable $F_\sigma^h = O(\lambda)$. Again, η_σ and F_σ^h are independent of t, u .

We are left with proving that $\eta_\sigma - \eta_\mu \neq 0$. It is sufficient to note that, by direct computation of the lowest order terms, for some $\vartheta > 0$, (6.1) can be written as:

$$\begin{aligned} z_h &= b_1 \lambda_h^2 + O(|\lambda|^2 \gamma^{\vartheta h}) + O(|\lambda|^3) \quad , \quad b_1 > 0 \\ s_h/\sigma_h &= -b_2 \lambda_h + O(|\lambda| \gamma^{\vartheta h}) + O(|\lambda|^2) \quad , \quad b_2 > 0 \\ m_h/\mu_h &= b_2 \lambda_h + O(|\lambda| \gamma^{\vartheta h}) + O(|\lambda|^2) \quad , \quad b_2 > 0 , \end{aligned} \quad (6.23)$$

where b_1, b_2 are constants independent of λ and h . Using (6.23) and the definitions of η_μ and η_σ we find: $\eta_\sigma - \eta_\mu = (2b_2/\log \gamma)\lambda + O(\lambda^2)$. ■

6.4. The scale h_1^*

The integration described in Chapter 5 is iterated until a scale h_1^* defined in the following way:

$$h_1^* \stackrel{def}{=} \begin{cases} \min \{1, [\log_\gamma |\sigma_1|^{\frac{1}{1-\eta_\sigma}}]\} & \text{if } |\sigma_1|^{\frac{1}{1-\eta_\sigma}} > 2|\mu_1|^{\frac{1}{1-\eta_\mu}}, \\ \min \{1, [\log_\gamma |u|^{\frac{1}{1-\eta_\mu}}]\} & \text{if } |\sigma_1|^{\frac{1}{1-\eta_\sigma}} \leq 2|\mu_1|^{\frac{1}{1-\eta_\mu}}. \end{cases} \quad (6.24)$$

From (6.24) it follows that

$$C_2 \gamma^{h_1^*} \leq |\sigma_{h_1^*}| + |\mu_{h_1^*}| \leq C_1 \gamma^{h_1^*}, \quad (6.25)$$

with C_1, C_2 independent of λ, μ_1, σ_1 .

This is obvious in the case $h_1^* = 1$. If $h_1^* < 1$ and $|\sigma_1|^{\frac{1}{1-\eta_\sigma}} > 2|\mu_1|^{\frac{1}{1-\eta_\mu}}$, then $\gamma^{h_1^*-1} = c_\sigma |\sigma_1|^{\frac{1}{1-\eta_\sigma}}$, with $1 \leq c_\sigma < \gamma$, so that, using the third of (6.19), we see that $C_2 \gamma^{h_1^*} \leq |\sigma_{h_1^*}| \leq C'_1 g^{h_1^*}$, for some $C'_1, C_2 = O(1)$. Furthermore, using also the second of (6.19), we find

$$\frac{|\mu_{h_1^*}|}{|\sigma_{h_1^*}|} = c_\sigma^{\eta_\sigma - \eta_\mu} |\mu_1| |\sigma_1|^{-\frac{1-\eta_\mu}{1-\eta_\sigma}} \gamma^{F_\mu^{h_1^*} - F_\sigma^{h_1^*}} < 1 \quad (6.26)$$

and (6.25) follows.

If $h_1^* < 1$ and $|\sigma_1|^{\frac{1}{1-\eta_\sigma}} \leq 2|\mu_1|^{\frac{1}{1-\eta_\mu}}$, then $\gamma^{h_1^*-1} = c_u |u|^{\frac{1}{1-\eta_\mu}}$, with $1 \leq c_u < \gamma$, so that, using the second of (6.19) and $|\mu_1| = O(|u|)$, we see that $C_2 \gamma^{h_1^*} \leq |\mu_{h_1^*}| \leq C'_1 \gamma^{h_1^*}$. Furthermore, using the third (6.19), we find

$$\frac{|\sigma_{h_1^*}|}{|\mu_{h_1^*}|} = c_u^{\eta_\sigma - \eta_\mu} |\sigma_1| |u|^{-\frac{1-\eta_\sigma}{1-\eta_\mu}} \gamma^{F_\sigma^{h_1^*} - F_\mu^{h_1^*}} < C''_1, \quad (6.27)$$

for some $C''_1 = O(1)$, and (6.25) again follows.

Remark. The specific value of h_1^* is not crucial: if we change h_1^* in $h_1^* + n$, $n \in \mathbb{Z}$, the constants C_1, C_2

in (6.25) are replaced by different $O(1)$ constants and the estimates below are not qualitatively modified. Of course, the specific values of C_1, C_2 (then, the specific value of h_1^*) can affect the convergence radius of the perturbative series in λ . The optimal value of h_1^* should be chosen by maximizing the corresponding convergence radius. Since here we are not interested in optimal estimates, we find the choice in (6.24) convenient.

Note also that h_1^* is a non analytic function of (λ, t, u) (in particular for small u we have $\gamma^{h_1^*} \sim |u|^{1+O(\lambda)}$). As a consequence, the asymptotic expression for the specific heat near the critical points (that we shall obtain in next section) will contain non analytic functions of u (in fact it will contain terms depending on h_1^*). However, as remarked after the Main Theorem in Chapter 1, this does not imply that C_v is non analytic: it is clear that in this case the non analyticity is introduced “by hands” by our specific choice of h_1^* .

From the results of Theorem 6.1 and Lemma 6.1, together with (6.24) and (6.25), it follows that the assumptions of Theorem 5.1 are satisfied for any $\bar{h} \geq h_1^*$. The integration of the scales $\leq h_1^*$ must be performed in a different way, as will be discussed in next Chapter.

7. Renormalization Group for light fermions. The non anomalous regime.

In the preceding Chapters, we have explained how to integrate the ψ fields up to the scale h_1^* , defined in the last section of previous Chapter. We managed to prove that, up to that scale, the running coupling constants can be bounded as in (5.22), (5.23) and (5.29), so that the iterative construction is inductively well defined, and the kernels of the effective potentials can be bounded, step by step, as stated by Theorem 5.1. Once we reach the scale h_1^* , the bound (5.23) stops to be true and the bounds leading to Theorem 5.1 fail. In particular the crucial bound (5.54) stops to be true. As a consequence, from this scale on, we have to proceed via a different iterative procedure. The idea is to use conditions (5.29), which hold with an equal sign on scale $\bar{h} = h_1^*$, to prove that the $\psi^{(\leq h_1^*)}$ field can be written, by a rotation which is essentially the inverse of (2.13), as a sum of two fields $\psi^{(1, \leq h_1^*)}$, $\psi^{(2, \leq h_1^*)}$, one of whom is massive on scale h_1^* (*i.e.* with mass $O(\gamma^{h_1^*})$). It is then easy to show that one can integrate in one step (*i.e.* without any further multiscale integration) the massive field, so that one is left with an effective theory involving only the (nearly) massless field.

The two fields correspond to the variables associated to the two original Ising layers. We can then say that on scale h_1^* the theory is effectively described by a theory of two interacting Ising layers, with (anomalously) renormalized parameters. On scale h_1^* one of the two layers (the one corresponding to the massive field) is well far from criticality and the corresponding variables can be easily integrated out; we are left with the theory of a single perturbed Ising model with renormalized parameters. The multiscale integration for the latter will be much easier than that described above, and in particular it will not involve any anomalous flow of the effective renormalization constants.

In the present Chapter we will first describe the integration of the massive field and the iterative integration of left over massless field. A corollary of the construction will be the analyticity of the free energy for temperatures different from the critical ones. Finally we will derive and solve the equation for the critical temperatures, leading to (1.7).

7.1. Integration of the $\psi^{(1)}$ fields

If h_1^* is fixed as in §6.4, we can use Theorem 5.1 up to the scale $\bar{h} = h_1^* + 1$.

Once that all the scales $> h_1^*$ are integrated out, it is more convenient to describe the system in terms of the fields $\psi_\omega^{(1)}, \psi_\omega^{(2)}$, $\omega = \pm 1$, defined through the following change of variables:

$$\hat{\psi}_{\omega, \mathbf{k}}^{\alpha(\leq h_1^*)} = \frac{1}{\sqrt{2}}(\hat{\psi}_{\omega, -\alpha \mathbf{k}}^{(1, \leq h_1^*)} - i\alpha \hat{\psi}_{\omega, -\alpha \mathbf{k}}^{(2, \leq h_1^*)}), \quad \psi_{\omega, \mathbf{x}}^{(j)} = \frac{1}{M^2} \sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}} \hat{\psi}_{\omega, \mathbf{k}}^{(j)}. \quad (7.1)$$

If we perform this change of variables, we find $P_{Z_{h_1^*}, \sigma_{h_1^*}, \mu_{h_1^*}, C_{h_1^*}} = \prod_{j=1}^2 P_{Z_{h_1^*}, m_{h_1^*}^{(j)}, C_{h_1^*}}^{(j)}$ where, if we define

$$\Psi_{\mathbf{k}}^{(j, \leq h_1^*), T \text{ def}} (\psi_{1, \mathbf{k}}^{(j, \leq h_1^*)}, \psi_{-1, \mathbf{k}}^{(j, \leq h_1^*)}),$$

$$\begin{aligned} & P_{Z_{h_1^*}, m_{h_1^*}^{(j)}, C_{h_1^*}}^{(j)} (d\psi^{(j, \leq h_1^*)}) \stackrel{\text{def}}{=} \\ & \stackrel{\text{def}}{=} \frac{1}{N_{h_1^*}^{(j)}} \prod_{\mathbf{k}, \omega} d\psi_{\omega, \mathbf{k}}^{(j, \leq h_1^*)} \exp \left\{ -\frac{Z_{h_1^*}}{4M^2} \sum_{\mathbf{k} \in D_{h_1^*}} C_{h_1^*}(\mathbf{k}) \Psi_{\mathbf{k}}^{(j, \leq h_1^*), T} A_j^{(h_1^*)}(\mathbf{k}) \Psi_{-\mathbf{k}}^{(j, \leq h_1^*)} \right\} \\ & A_j^{(h_1^*)}(\mathbf{k}) \stackrel{\text{def}}{=} \begin{pmatrix} (-i \sin k - \sin k_0) + a_{h_1^*}^{+(j)}(\mathbf{k}) & -i(m_{h_1^*}^{(j)}(\mathbf{k}) + c_{h_1^*}^{(j)}(\mathbf{k})) \\ i(m_{h_1^*}^{(j)}(\mathbf{k}) + c_{h_1^*}^{(j)}(\mathbf{k})) & (-i \sin k + \sin k_0) + a_{h_1^*}^{-(j)}(\mathbf{k}) \end{pmatrix} \end{aligned} \quad (7.2)$$

and $a_{h_1^*}^{\omega(j)}$, $m_{h_1^*}^{(j)}$, $c_{h_1^*}^{(j)}$ are given by (A4.2) with $h = h^* + 1$.

The propagators $g_{\omega_1, \omega_2}^{(j, \leq h_1^*)}$ associated with the fermionic integration (7.2) are given by (A4.1) with $h = h_1^* + 1$. Note that, by (6.25), $\max\{|m_{h_1^*}^{(1)}|, |m_{h_1^*}^{(2)}|\} = |\sigma_{h_1^*}| + |\mu_{h_1^*}| = O(\gamma^{h_1^*})$ (see (A4.2) for the definition of $m_{h_1^*}^{(1)}$, $m_{h_1^*}^{(2)}$). From now on, for definiteness we shall suppose that $\max\{|m_{h_1^*}^{(1)}|, |m_{h_1^*}^{(2)}|\} \equiv |m_{h_1^*}^{(1)}|$. Then, it is easy to realize that the propagator $g_{\omega_1, \omega_2}^{(1, \leq h_1^*)}$ is bounded as follows.

$$|\partial_{x_0}^{n_0} \partial_x^{n_1} g_{\omega_1, \omega_2}^{(1, \leq h_1^*)}(\mathbf{x})| \leq C_{N, n} \frac{\gamma^{(1+n)h_1^*}}{1 + (\gamma^{h_1^*} |\mathbf{d}(\mathbf{x})|)^N} \quad , \quad n = n_0 + n_1 \quad , \quad (7.3)$$

namely $g_{\omega_1, \omega_2}^{(1, \leq h_1^*)}$ satisfies the same bound as the single scale propagator on scale $h = h_1^*$. This suggests to integrate out $\psi^{(1, \leq h_1^*)}$, without any other scale decomposition. We find the following result.

LEMMA 7.1 *If $|\lambda| \leq \varepsilon_1$, $|\sigma_1|, |\mu_1| \leq c_1$ (c_1, ε_1 being the same as in Theorem 4.1) and ν_1 is fixed as in Theorem 6.1, we can rewrite the partition function as*

$$\Xi_{AT}^- = \int P_{Z_{h_1^*}, \widehat{m}_{h_1^*}^{(2)}, C_{h_1^*}}^{(2)} (d\psi^{(2, \leq h_1^*)}) e^{-\overline{\mathcal{V}}^{(h_1^*)}(\sqrt{Z_{h_1^*}} \psi^{(2, \leq h_1^*)}) - M^2 \overline{E}_{h_1^*}} \quad , \quad (7.4)$$

where: $\widehat{m}_{h_1^*}^{(2)}(\mathbf{k}) = m_{h_1^*}^{(2)}(\mathbf{k}) - \gamma^{h_1^*} \pi_{h_1^*} C_{h_1^*}^{-1}(\mathbf{k})$, with $\pi_{h_1^*}$ a free parameter, s.t. $|\pi_{h_1^*}| \leq c|\lambda|$; $|\overline{E}_{h_1^*} - E_{h_1^*}| \leq c|\lambda|\gamma^{2h_1^*}$; and

$$\begin{aligned} \overline{\mathcal{V}}^{(h_1^*)}(\psi^{(2)}) - \gamma^{h_1^*} \pi_{h_1^*} F_{\sigma}^{(2, \leq h_1^*)}(\psi^{(2, \leq h_1^*)}) &= \sum_{n=1}^{\infty} \sum_{\underline{\omega}} \prod_{i=1}^{2n} \hat{\psi}_{\omega_i, \mathbf{k}_i}^{(2)} \overline{W}_{2n, \underline{\omega}}^{(h_1^*)}(\mathbf{k}_1, \dots, \mathbf{k}_{2n-1}) \delta\left(\sum_{i=1}^{2n} \mathbf{k}_i\right) = \\ &= \sum_{n=1}^{\infty} \sum_{\underline{\sigma}, \underline{j}, \underline{\omega}} \prod_{i=1}^{2n} \partial_{j_i}^{\sigma_i} \psi_{\omega_i, \mathbf{x}_i}^{(2)} \overline{W}_{2n, \underline{\sigma}, \underline{j}, \underline{\omega}}^{(h_1^*)}(\mathbf{x}_1, \dots, \mathbf{x}_{2n}) \quad , \end{aligned} \quad (7.5)$$

with $F_{\sigma}^{(2, \leq h)}$ given by the first of (5.10) with $\hat{\psi}_{\omega, \mathbf{k}}^{(2, \leq h)} \hat{\psi}_{\omega', -\mathbf{k}}^{(2, \leq h)}$ replacing $\hat{\psi}_{\omega, \mathbf{k}}^{+(\leq h)} \hat{\psi}_{\omega', \mathbf{k}}^{-(\leq h)}$; and $\overline{W}_{2n, \underline{\sigma}, \underline{j}, \underline{\omega}}^{(h_1^*)}$ satisfying the same bound (5.30) as $W_{2n, \underline{\sigma}, \underline{j}, \underline{\omega}}^{(\bar{h})}$ with $\bar{h} = h_1^*$.

In order to prove the Lemma it is sufficient to consider (5.2) with $h = h_1^*$ and rewrite $P_{Z_{h_1^*}, \sigma_{h_1^*}, \mu_{h_1^*}, C_{h_1^*}}$ as the product $\prod_{j=1}^2 P_{Z_{h_1^*}, m_{h_1^*}^{(j)}, C_{h_1^*}}^{(j)}$. Then the integration over the $\psi^{(1, \leq h_1^*)}$ field is done as the integration of the χ 's in Chapter 4, recalling the bound (7.3).

Finally we rewrite $m_{h_1^*}^{(2)}(\mathbf{k})$ as $\widehat{m}_{h_1^*}^{(2)}(\mathbf{k}) + \gamma^{h_1^*} \pi_{h_1^*} C_{h_1^*}^{-1}(\mathbf{k})$, where $\pi_{h_1^*}$ is a parameter to be suitably fixed below as a function of λ, σ_1, μ_1 .

7.2. The localization operator

The integration of the r.h.s. of (7.4) is done in an iterative way similar to the one described in Chapter 5. If $h = h_1^*, h_1^* - 1, \dots$, we shall write:

$$\Xi_{AT}^- = \int P_{Z_h, \widehat{m}_h^{(2)}, C_h}^{(2)} (d\psi^{(2, \leq h)}) e^{-\overline{\mathcal{V}}^{(h)}(\sqrt{Z_h} \psi^{(2, \leq h)}) - M^2 E_h} \quad , \quad (7.6)$$

where $\overline{\mathcal{V}}^{(h)}$ is given by an expansion similar to (5.39), with h replacing h_1^* and $Z_h, \widehat{m}_h^{(2)}$ are defined recursively in the following way. We first introduce a *localization operator* \mathcal{L} . As in §5.2, we define \mathcal{L} as a combination of four operators \mathcal{L}_j and $\overline{\mathcal{P}}_j$, $j = 0, 1$. \mathcal{L}_j are defined as in (5.6) and (5.7), while $\overline{\mathcal{P}}_0$ and $\overline{\mathcal{P}}_1$, in analogy

with (5.8), are defined as the operators extracting from a functional of $\widehat{m}_h^{(2)}(\mathbf{k})$, $h \leq h_1^*$, the contributions independent and linear in $\widehat{m}_h^{(2)}(\mathbf{k})$. Note that inductively the kernels $\overline{W}_{2n,\underline{\omega}}^{(h)}$ can be thought as functionals of $\widehat{m}_k(\mathbf{k})$, $h \leq k \leq h_1^*$. Given $\mathcal{L}_j, \overline{\mathcal{P}}_j$, $j = 0, 1$ as above, we define the action of \mathcal{L} on the kernels $\overline{W}_{2n,\underline{\omega}}^{(h)}$ as follows.

1) If $n = 1$, then

$$\mathcal{L}\overline{W}_{2,\underline{\omega}}^{(h)} \stackrel{def}{=} \begin{cases} \mathcal{L}_0(\overline{\mathcal{P}}_0 + \overline{\mathcal{P}}_1)\overline{W}_{2,\underline{\omega}}^{(h)} & \text{if } \omega_1 + \omega_2 = 0, \\ \mathcal{L}_1\overline{\mathcal{P}}_0\overline{W}_{2,\underline{\omega}}^{(h)} & \text{if } \omega_1 + \omega_2 \neq 0. \end{cases}$$

2) If $n > 2$, then $\mathcal{L}\overline{W}_{2n,\underline{\omega}}^{(h)} = 0$.

It is easy to prove the analogue of Lemma 5.1:

$$\mathcal{L}\overline{\mathcal{V}}^{(h)} = (s_h + \gamma^h p_h)F_\sigma^{(2,\leq h)} + z_h F_\zeta^{(2,\leq h)}, \quad (7.7)$$

where s_h, p_h and z_h are real constants and: s_h is linear in $\widehat{m}_k^{(2)}(\mathbf{k})$, $h \leq k \leq h_1^*$; p_h and z_h are independent of $\widehat{m}_k^{(2)}(\mathbf{k})$. Furthermore $F_\sigma^{(2,\leq h)}$ and $F_\zeta^{(2,\leq h)}$ are given by the first and the last of (5.10) with $\hat{\psi}_{\omega,\mathbf{k}}^{(2,\leq h)} \hat{\psi}_{\omega',-\mathbf{k}}^{(2,\leq h)}$ replacing $\hat{\psi}_{\omega,\mathbf{k}}^{+(\leq h)} \hat{\psi}_{\omega',\mathbf{k}}^{-(\leq h)}$.

Remark. Note that the action of \mathcal{L} on the quartic terms is trivial. The reason of such a choice is that in the present case no quartic local term can appear, because of Pauli principle: $\psi_{1,\mathbf{x}}^{(2,h)} \psi_{1,\mathbf{x}}^{(2,h)} \psi_{-1,\mathbf{x}}^{(2,h)} \psi_{-1,\mathbf{x}}^{(2,h)} \equiv 0$, so that $\mathcal{L}_0\overline{W}_{4,\underline{\omega}} = 0$.

Using the symmetry properties exposed in §4.3, we can prove the analogue of Lemma 5.2: if $n = 1$, then

$$\mathcal{R}\overline{W}_{2,\underline{\omega}} = \begin{cases} [\overline{\mathcal{S}}_2 + \mathcal{R}_2(\overline{\mathcal{P}}_0 + \overline{\mathcal{P}}_1)]\overline{W}_{2,\underline{\omega}} & \text{if } \omega_1 + \omega_2 = 0, \\ [\mathcal{R}_1\overline{\mathcal{S}}_1 + \mathcal{R}_2\overline{\mathcal{P}}_0]\overline{W}_{2,\underline{\omega}} & \text{if } \omega_1 + \omega_2 \neq 0, \end{cases} \quad (7.8)$$

where $\overline{\mathcal{S}}_1 = 1 - \overline{\mathcal{P}}_0$ and $\overline{\mathcal{S}}_2 = 1 - \overline{\mathcal{P}}_0 - \overline{\mathcal{P}}_1$; if $n = 2$, then $\overline{W}_{4,\underline{\omega}} = \mathcal{R}_1\overline{W}_{4,\underline{\omega}}$.

7.3. Renormalization for $h \leq h_1^*$

If \mathcal{L} and $\mathcal{R} = 1 - \mathcal{L}$ are defined as in previous subsection, we can rewrite (7.6) as:

$$\int P_{Z_h, \widehat{m}_h^{(2)}, C_h}^{(2)}(d\psi^{(2,\leq h)}) e^{-\mathcal{L}\overline{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(2,\leq h)}) - \mathcal{R}\overline{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(2,\leq h)}) - M^2 E_h}. \quad (7.9)$$

Furthermore, using (7.7) and defining:

$$\widehat{Z}_{h-1}(\mathbf{k}) \stackrel{def}{=} Z_h(1 + C_h^{-1}(\mathbf{k})z_h) \quad , \quad \widehat{m}_{h-1}^{(2)}(\mathbf{k}) \stackrel{def}{=} \frac{Z_h}{\widehat{Z}_{h-1}(\mathbf{k})} \left(\widehat{m}_h^{(2)}(\mathbf{k}) + C_h^{-1}(\mathbf{k})s_h \right), \quad (7.10)$$

we see that (7.9) is equal to

$$\int P_{\widehat{Z}_{h-1}, \widehat{m}_{h-1}^{(2)}, C_h}^{(2)}(d\psi^{(2,\leq h)}) e^{-\gamma^h p_h F_\sigma^{(2,\leq h)}(\sqrt{Z_h}\psi^{(2,\leq h)}) - \mathcal{R}\overline{\mathcal{V}}^h(\sqrt{Z_h}\psi^{(2,\leq h)}) - M^2(E_h + t_h)} \quad (7.11)$$

Again, we rescale the potential:

$$\widetilde{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}) \stackrel{def}{=} \gamma^h \pi_h F_\sigma^{(2,\leq h)}(\sqrt{Z_{h-1}}\psi^{(2,\leq h)}) + \mathcal{R}\overline{\mathcal{V}}^h(\sqrt{Z_h}\psi^{(2,\leq h)}), \quad (7.12)$$

where $Z_{h-1} = \hat{Z}_{h-1}(\mathbf{0})$ and $\pi_h = (Z_h/Z_{h-1})p_h$; we define \tilde{f}_h^{-1} as in (5.18), we perform the single scale integration and we define the new effective potential as

$$e^{-\bar{\mathcal{V}}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(2,\leq h-1)})-M^2\tilde{E}_h} \stackrel{def}{=} \int P_{Z_{h-1},\hat{m}_{h-1}^{(2)},\tilde{f}_h^{-1}}^{(2)}(d\psi^{(2,h)})e^{-\tilde{\mathcal{V}}^h(\sqrt{Z_h}\psi^{(2,\leq h)})}. \quad (7.13)$$

Finally we pose $E_{h-1} = E_h + t_h + \tilde{E}_h$. Note that the above procedure allow us to write the π_h in terms of π_k , $h \leq k \leq h_1^*$, namely $\pi_{h-1} = \gamma^h \pi_h + \beta_\pi^h(\pi_h, \dots, \pi_{h_1^*})$, where β_π^h is the *Beta function*.

Proceeding as in §4 we can inductively show that $\bar{\mathcal{V}}^{(h)}$ has the structure of (7.5), with h replacing h_1^* and that the kernels of $\bar{\mathcal{V}}^{(h)}$ are bounded as follows.

LEMMA 7.2. *Let the hypothesis of Lemma 5.1 be satisfied and suppose that, for $\bar{h} < h \leq h_1^*$ and some constants $c, \vartheta > 0$*

$$e^{-c|\lambda|} \leq \frac{\hat{m}_h^{(2)}}{\hat{m}_{h-1}^{(2)}} \leq e^{c|\lambda|} \quad , \quad e^{-c|\lambda|^2} \leq \frac{Z_h}{Z_{h-1}} \leq e^{c|\lambda|^2} \quad , \quad |\pi_h| \leq c|\lambda| \quad , \quad |\hat{m}_h^{(2)}| \leq \gamma^{\bar{h}}. \quad (7.14)$$

Then the partition function can be rewritten as in (7.6) and there exists $C > 0$ s.t. the kernels of $\bar{\mathcal{V}}^{(h)}$ satisfy:

$$\int d\mathbf{x}_1 \cdots d\mathbf{x}_{2n} |\overline{W}_{2n,\underline{\sigma},\underline{j},\underline{\omega}}^{(\bar{h})}(\mathbf{x}_1, \dots, \mathbf{x}_{2n})| \leq M^2 \gamma^{-\bar{h}D_k(n)} (C|\lambda|)^{\max(1,n-1)} \quad (7.15)$$

where $D_k(n) = -2 + n + k$ and $k = \sum_{i=1}^{2n} \sigma_i$. Finally $|E_{\bar{h}+1}| + |t_{\bar{h}+1}| \leq c|\lambda|\gamma^{2\bar{h}}$.

The proof of Lemma 7.2 is essentially identical to the proof of Theorem 5.1 and we do not repeat it here.

It is possible to fix $\pi_{h_1^*}$ so that the first three assumptions in (7.14) are valid for any $h \leq h_1^*$. More precisely, the following result holds, see Appendix A8 for the proof.

LEMMA 7.3. *If $|\lambda| \leq \varepsilon_1$, $|\sigma_1|, |\mu_1| \leq c_1$ and ν_1 is fixed as in Theorem 4.1, there exists $\pi_{h_1^*}^*(\lambda, \sigma_1, \mu_1)$ such that, if we fix $\pi_{h_1^*} = \pi_{h_1^*}^*(\lambda, \sigma_1, \mu_1)$, for $h \leq h_1^*$ we have:*

$$|\pi_h| \leq c|\lambda|\gamma^{(\vartheta/2)(h-h_1^*)} \quad , \quad \hat{m}_h^{(2)} = \hat{m}_{h_1^*}^{(2)}\gamma^{F_m^h} \quad , \quad Z_h = Z_{h_1^*}\gamma^{\overline{F}_\zeta^h}, \quad (7.16)$$

where F_m^h and \overline{F}_ζ^h are $O(\lambda)$. Moreover:

$$\left| \pi_{h_1^*}^*(\lambda, \sigma_1, \mu_1) - \pi_{h_1^*}^*(\lambda, \sigma'_1, \mu'_1) \right| \leq c|\lambda| \left(\gamma^{(\eta_\sigma-1)h_1^*} |\sigma_1 - \sigma'_1| + \gamma^{(\eta_\mu-1)h_1^*} |\mu_1 - \mu'_1| \right). \quad (7.17)$$

7.4. The integration of the scales $\leq h_2^*$

In order to insure that the last assumption in (7.14) holds, we iterate the preceding construction up to the scale h_2^* defined as the scale s.t. $|\hat{m}_k^{(2)}| \leq \gamma^{k-1}$ for any $h_2^* \leq k \leq h_1^*$ and $|\hat{m}_{h_2^*-1}^{(2)}| > \gamma^{h_2^*-2}$.

Once we have integrated all the fields $\psi^{(>h_2^*)}$, we can integrate $\psi^{(2,\leq h_2^*)}$ without any further multiscale decomposition. Note in fact that by definition the propagator satisfies the same bound (7.3) with h_2^* replacing h_1^* . Then, if we define

$$e^{-M^2\tilde{E}_{\leq h_2^*} \stackrel{def}{=} \int P_{Z_{h_2^*-1},\hat{m}_{h_2^*-1}^{(2)},C_{h_2^*}} e^{-\tilde{\mathcal{V}}^{(h_2^*)}(\sqrt{Z_{h_2^*-1}}\psi^{(2,\leq h_2^*)})}, \quad (7.18)$$

we find that $|\tilde{E}_{\leq h_2^*}| \leq c|\lambda|\gamma^{2h_2^*}$ (the proof is a repetition of the estimates on the single scale integration).

Combining this bound with the results of Theorem 5.1, Lemma 7.1, Lemma 7.2 and Lemma 7.3, together with the results of Chapter 5, we finally find that the free energy associated to Ξ_{AT}^- is given by the following *finite* sum, uniformly convergent with the size of Λ_M :

$$\lim_{M \rightarrow \infty} \frac{1}{M^2} \log \Xi_{AT}^- = E_{\leq h_2^*} + (\bar{E}_{h_1^*} - E_{h_1^*}) + \sum_{h=h_2^*+1}^1 (\tilde{E}_h + t_h), \quad (7.19)$$

where $E_{\leq h_2^*} = \lim_{M \rightarrow \infty} \tilde{E}_{\leq h_2^*}$ and it is easy to see that $E_{\leq h_2^*}$, for any finite h_2^* , exists and satisfies the same bound of $\tilde{E}_{h_2^*}$.

7.5. Keeping h_2^* finite.

From the discussion of previous subsection, it follows that, for any finite h_2^* , (7.19) is an analytic function of λ, t, u , for $|\lambda|$ sufficiently small, uniformly in h_2^* (this is an elementary consequence of Vitali's convergence theorem). Moreover, in Appendix A9 it is proved that, for any $\gamma^{h_2^*} > 0$, the limit (7.19) coincides with $\lim_{M \rightarrow \infty} 1/M^2 \log \Xi_{AT}^{\gamma_1, \gamma_2}$ for any choice γ_1, γ_2 of boundary conditions; hence this limit coincides with $-2 \log(2 \cosh \beta \lambda)$ plus the free energy in (1.3). We can state the result as follows.

LEMMA 7.4. *There exists $\varepsilon_1 > 0$ such that, if $|\lambda| \leq \varepsilon_1$ and $t \pm u \in D$ (the same as in the Main Theorem in Chapter 1), the free energy f defined in (1.3) is real analytic in λ, t, u , except possibly for the choices of λ, t, u such that $\gamma^{h_2^*} = 0$.*

We shall see in next Chapter that the specific heat is logarithmically divergent as $\gamma^{h_2^*} \rightarrow 0$. So the critical point is really given by the condition $\gamma^{h_2^*} = 0$. We shall explicitly solve the equation for the critical point in next subsection.

7.6. The critical points.

In the present subsection we check that, if $t \pm u \in D$, D being a suitable interval centered around $\sqrt{2} - 1$, see Main Theorem in Chapter 1, there are precisely two critical points, of the form (1.7). More precisely, keeping in mind that the equation for the critical point is simply $\gamma^{h_2^*} = 0$ (see the end of previous subsection), we prove the following.

LEMMA 7.5. *Let $|\lambda| \leq \varepsilon_1$, $t \pm u \in D$ and $\pi_{h_1^*}$ be fixed as in Lemma 7.3. Then $\gamma^{h_2^*} = 0$ only if $(\lambda, t, u) = (\lambda, t_c^\pm(\lambda, u), u)$, where $t_c^\pm(\lambda, u)$ is given by (1.7).*

PROOF - From the definition of h_2^* given above, see §7.4, it follows that h_2^* satisfies the following equation:

$$\gamma^{h_2^*-1} = c_m \gamma^{F_m^{h_2^*}} \left| |\sigma_{h_1^*}| - |\mu_{h_1^*}| - \alpha_\sigma \gamma^{h_1^*} \pi_{h_1^*} \right|, \quad (7.20)$$

for some $1 \leq c_m < \gamma$ and $\alpha_\sigma = \text{sign } \sigma_1$. Then, the equation $\gamma^{h_2^*} = 0$ can be rewritten as:

$$|\sigma_{h_1^*}| - |\mu_{h_1^*}| - \alpha_\sigma \gamma^{h_1^*} \pi_{h_1^*} = 0. \quad (7.21)$$

First note that the result of Lemma 7.5 is trivial when $h_1^* = 1$. If $h_1^* < 1$, (7.21) cannot be solved when $|\sigma_1|^{\frac{1}{1-\eta_\sigma}} > 2|\mu_1|^{\frac{1}{1-\eta_\mu}}$. In fact,

$$\begin{aligned} & |\sigma_1| \gamma^{\eta_\sigma(h_1^*-1)+F_\sigma^{h_1^*}} - |\mu_1| \gamma^{\eta_\mu(h_1^*-1)+F_\mu^{h_1^*}} - \alpha_\sigma \gamma^{h_1^*} \pi_{h_1^*} = \\ & = |\sigma_1|^{1+\frac{\eta_\sigma}{1-\eta_\sigma}} c_1 - \left(|\mu_1| |\sigma_1|^{-\frac{1-\eta_\mu}{1-\eta_\sigma}} \right) |\sigma_1|^{\frac{1-\eta_\mu}{1-\eta_\sigma} - \frac{\eta_\mu}{1-\eta_\sigma}} c_1' - \alpha_\sigma \gamma^{h_1^*} \pi_{h_1^*} \geq \frac{\gamma^{h_1^*-1}}{3\gamma}, \end{aligned} \quad (7.22)$$

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where c_1, c'_1 are constants $= 1 + O(\lambda)$, $\pi_{h_1^*} = O(\lambda)$ and $\gamma^{h_1^*-1} = c_\sigma |\sigma_1|^{\frac{1}{1-\eta_\sigma}}$, with $1 \leq c_\sigma < \gamma$. Now, if $|\mu_1| > 0$, the r.h.s. of (7.22) equation is strictly positive.

So, let us consider the case $h_1^* < 1$ and $|\sigma_1|^{\frac{1}{1-\eta_\sigma}} \leq 2|\mu_1|^{\frac{1}{1-\eta_\mu}}$ (s.t. $\gamma^{h_1^*} = c_u \log_\gamma |u|^{\frac{1}{1-\eta_\mu}}$, with $1 \leq c_u \leq \gamma$). In this case (7.21) can be easily solved to find:

$$|\sigma_1| = |\mu_1| |u|^{\frac{\eta_\mu - \eta_\sigma}{1-\eta_\mu}} c_u^{\eta_\mu - \eta_\sigma} \gamma^{F_\mu^{h_1^*} - F_\sigma^{h_1^*}} + |u|^{\frac{1-\eta_\sigma}{1-\eta_\mu}} c_u^{1-\eta_\sigma} \alpha_\sigma \gamma^{1-F_\sigma^{h_1^*}} \pi_{h_1^*}. \quad (7.23)$$

Note that $c_u^{\eta_\mu - \eta_\sigma} \gamma^{F_\mu^{h_1^*} - F_\sigma^{h_1^*}} = 1 + O(\lambda)$ is just a function of u , (it does not depend on t), because of our definition of h_1^* . Moreover $\pi_{h_1^*}$ is a smooth function of t : if we call $\pi_{h_1^*}(t, u)$ resp. $\pi_{h_1^*}(t', u)$ the correction corresponding to the initial data $\sigma_1(t, u), \mu_1(t, u)$ resp. $\sigma_1(t', u), \mu_1(t', u)$, we have

$$|\pi_{h_1^*}(t, u) - \pi_{h_1^*}(t', u)| \leq c|\lambda| |u|^{\frac{\eta_\sigma - 1}{1-\eta_\mu}} |t - t'|, \quad (7.24)$$

where we used (7.17) and the bounds $|\sigma_1 - \sigma'_1| \leq c|t - t'|$ and $|\mu_1 - \mu'_1| \leq c|u||t - t'|$, following from the definitions of (σ_1, μ_1) in terms of (σ, μ) and of (t, u) , see Chapter 4.

Using the same definitions we also realize that (7.23) can be rewritten as

$$t = \left[\sqrt{2} - 1 + \frac{\nu(\lambda)}{2} \pm |u|^{1+\eta} (1 + \lambda f(t, u)) \right] \frac{1 + \hat{\lambda}(t^2 - u^2)}{1 + \hat{\lambda}}, \quad (7.25)$$

where

$$1 + \eta \stackrel{\text{def}}{=} \frac{1 - \eta_\sigma}{1 - \eta_\mu}, \quad (7.26)$$

and the crucial property is that $\eta = -b\lambda + O(\lambda^2)$, $b > 0$, see Lemma 6.1 and (6.23). We also recall that both η and ν are functions of λ and are independent of t, u . Moreover $f(t, u)$ is a suitable bounded function s.t. $|f(t, u) - f(t', u)| \leq c|u|^{-(1+\eta)} |t - t'|$, as it follows from the Lipschitz property of $\pi_{h_1^*}$ (7.24). The r.h.s. of (7.25) is Lipschitz in t with constant $O(\lambda)$, so that (7.25) can be inverted w.r.t. t by contractions and, for both choices of the sign, we find a unique solution

$$t = t_c^\pm(\lambda, u) = \sqrt{2} - 1 + \nu^*(\lambda) \pm |u|^{1+\eta} (1 + F^\pm(\lambda, u)), \quad (7.27)$$

with $|F^\pm(\lambda, u)| \leq c|\lambda|$, for some c . ■

7.7. Computation of h_2^* .

Let us now solve (7.20) in the general case of $\gamma^{h_2^*} \geq 0$. Calling $\varepsilon \stackrel{\text{def}}{=} \gamma^{h_2^* - h_1^* - F_m^{h_2^*}} / c_m$, we find:

$$\begin{aligned} \varepsilon &= \left| |\sigma_1| \gamma^{(\eta_\sigma - 1)(h_1^* - 1) + F_\sigma^{h_1^*}} - |\mu_1| \gamma^{(\eta_\mu - 1)(h_1^* - 1) + F_\mu^{h_1^*}} - \alpha_\sigma \gamma \pi_{h_1^*} \right| = \\ &= \gamma^{(\eta_\sigma - 1)(h_1^* - 1) + F_\sigma^{h_1^*}} \left| |\sigma_1| - |\mu_1| \gamma^{(\eta_\mu - \eta_\sigma)(h_1^* - 1) + F_\mu^{h_1^*} - F_\sigma^{h_1^*}} - \alpha_\sigma \gamma^{1 + (1 - \eta_\sigma)(h_1^* - 1) - F_\sigma^{h_1^*}} \pi_{h_1^*} \right|. \end{aligned} \quad (7.28)$$

If $|\sigma_1|^{1/(1-\eta_\sigma)} \leq 2|\mu_1|^{1/(1-\eta_\mu)}$, we use $\gamma^{h_1^*-1} = c_u |u|^{1/(1-\eta_\mu)}$ and, from the second row of (7.27), we find: $\varepsilon = C \left| |\sigma_1| - |\sigma_{1,c}^{\alpha_\sigma}| \right| |u|^{-(1+\eta)}$, where $\sigma_{1,c}^\pm = \sigma_1(\lambda, t_c^\pm, u)$ and $C = C(\lambda, t, u)$ is bounded above and below by $O(1)$ constants; defining Δ as in (1.10), we can rewrite:

$$\varepsilon = C \frac{||\sigma_1| - |\sigma_{1,c}^{\alpha_\sigma}||}{|u|^{1+\eta}} = C' \frac{|\sigma_1^2 - (\sigma_{1,c}^{\alpha_\sigma})^2|}{\Delta |u|^{1+\eta}} = C'' \frac{|t - t_c^+| \cdot |t - t_c^-|}{\Delta^2}, \quad (7.29)$$

where $C' = C'(\lambda, t, u)$ and $C'' = C''(\lambda, t, u)$ are bounded above and below by $O(1)$ constants.

In the opposite case ($|\sigma_1|^{1/(1-\eta_s)} > 2|\mu_1|^{1/(1-\eta_\mu)}$), we use $\gamma^{h_1^*-1} = c_\sigma |\sigma_1|^{1/(1-\eta_\sigma)}$ and, from the first row of (7.27), we find $\varepsilon = \tilde{C}(1 - |\mu_1| |\sigma_1|^{-1/(1+\eta)} - \alpha_\sigma \gamma \pi_{h_1^*}) = \bar{C}$, where \tilde{C} and \bar{C} are bounded above and below by $O(1)$ constants. Since in this region of parameters $|t - t_c^\pm| \Delta^{-1}$ is also bounded above and below by $O(1)$ constants, we can in both cases write

$$\varepsilon = C_\varepsilon(\lambda, t, u) \frac{|t - t_c^+| \cdot |t - t_c^-|}{\Delta^2} \quad , \quad C_{1,\varepsilon} \leq C_\varepsilon(\lambda, t, u) \leq C_{2,\varepsilon} \quad (7.30)$$

and $C_{j,\varepsilon}$, $j = 1, 2$, are suitable positive $O(1)$ constants.

8. The specific heat.

In this Chapter we describe the expansion for the energy–energy correlation functions, from which we can derive a convergent expansion for the specific heat of the Ashkin–Teller model. We then compute the leading order contributing to the specific heat, we derive the expression (1.8), so concluding the proof of the Main Theorem in the Introduction.

Consider the specific heat defined in (1.3). The correlation function $\langle H_{\mathbf{x}}^{AT} H_{\mathbf{y}}^{AT} \rangle_{\Lambda_M, T}$ can be conveniently written as

$$\langle H_{\mathbf{x}}^{AT} H_{\mathbf{y}}^{AT} \rangle_{\Lambda, T} = \frac{\partial^2}{\partial \phi_{\mathbf{x}} \partial \phi_{\mathbf{y}}} \log \Xi_{AT}(\phi) \Big|_{\phi=0}, \quad \Xi_{AT}(\phi) \stackrel{def}{=} \sum_{\sigma^{(1)}, \sigma^{(2)}} e^{-\sum_{\mathbf{x} \in \Lambda} (1 + \phi_{\mathbf{x}}) H_{\mathbf{x}}^{AT}} \quad (8.1)$$

where $\phi_{\mathbf{x}}$ is a real commuting auxiliary field (with periodic boundary conditions).

Repeating the construction of Chapter 3, we see that $\Xi_{AT}(\phi)$ admit a Grassmanian representation similar to the one of Ξ_{AT} , and in particular, if $\mathbf{x} \neq \mathbf{y}$:

$$\begin{aligned} \frac{\partial^2}{\partial \phi_{\mathbf{x}} \partial \phi_{\mathbf{y}}} \log \Xi_{AT}(\phi) \Big|_{\phi=0} &= \frac{\partial^2}{\partial \phi_{\mathbf{x}} \partial \phi_{\mathbf{y}}} \log \sum_{\gamma_1, \gamma_2} (-1)^{\delta_{\gamma_1} + \delta_{\gamma_2}} \widehat{\Xi}_{AT}^{\gamma_1, \gamma_2}(\phi) \Big|_{\phi=0} \\ \widehat{\Xi}_{AT}^{\gamma_1, \gamma_2}(\phi) &= \int \prod_{\mathbf{x} \in \Lambda_M}^{j=1,2} dH_{\mathbf{x}}^{(j)} d\overline{H}_{\mathbf{x}}^{(j)} dV_{\mathbf{x}}^{(j)} d\overline{V}_{\mathbf{x}}^{(j)} e^{S_{\gamma_1}^{(1)}(t^{(1)}) + S_{\gamma_2}^{(2)}(t^{(2)}) + V_{\lambda} + \mathcal{B}(\phi)} \end{aligned} \quad (8.2)$$

where δ_{γ} , $S^{(j)}(t^{(j)})$ and V_{λ} where defined in Chapter 3, the apex γ_1, γ_2 attached to $\widehat{\Xi}_{AT}$ refers to the boundary conditions assigned to the Grassmanian fields and finally $\mathcal{B}(\phi)$ is defined as:

$$\begin{aligned} \mathcal{B}(\phi) &= \sum_{\mathbf{x} \in \Lambda} \phi_{\mathbf{x}} \left\{ a^{(1)} (\overline{H}_{\mathbf{x}}^{(1)} H_{\mathbf{x}+\hat{e}_1}^{(1)} + \overline{V}_{\mathbf{x}}^{(1)} V_{\mathbf{x}+\hat{e}_0}^{(1)}) + a^{(2)} (\overline{H}_{\mathbf{x}}^{(2)} H_{\mathbf{x}+\hat{e}_1}^{(2)} + \overline{V}_{\mathbf{x}}^{(2)} V_{\mathbf{x}+\hat{e}_0}^{(2)}) + \right. \\ &\quad \left. + \lambda \tilde{a} (\overline{H}_{\mathbf{x}}^{(1)} H_{\mathbf{x}+\hat{e}_1}^{(1)} \overline{H}_{\mathbf{x}}^{(2)} H_{\mathbf{x}+\hat{e}_1}^{(2)} + \overline{V}_{\mathbf{x}}^{(1)} V_{\mathbf{x}+\hat{e}_0}^{(1)} \overline{V}_{\mathbf{x}}^{(2)} V_{\mathbf{x}+\hat{e}_0}^{(2)}) \right\} \stackrel{def}{=} \sum_{\mathbf{x} \in \Lambda} \phi_{\mathbf{x}} A_{\mathbf{x}}, \end{aligned} \quad (8.3)$$

where $a^{(1)}$, $a^{(2)}$ and \tilde{a} are $O(1)$ constants, with $a^{(1)} - a^{(2)} = O(u)$. Using (8.2) and (8.3) we can rewrite:

$$\langle H_{\mathbf{x}}^{AT} H_{\mathbf{y}}^{AT} \rangle_{\Lambda, T} = \frac{1}{4} (\cosh J)^{2M^2} \sum_{\gamma_1, \gamma_2} (-1)^{\delta_{\gamma_1} + \delta_{\gamma_2}} \frac{\Xi_{AT}^{\gamma_1, \gamma_2}}{\Xi_{AT}} \langle A_{\mathbf{x}} A_{\mathbf{y}} \rangle_{\Lambda_M, T}^{\gamma_1, \gamma_2}, \quad (8.4)$$

where $\langle \cdot \rangle_{\Lambda_M, T}^{\gamma_1, \gamma_2}$ is the average w.r.t. the boundary conditions γ_1, γ_2 . Proceeding as in Appendix A9 one can show that, if $\gamma^{h_2^*} > 0$, $\langle A_{\mathbf{x}} A_{\mathbf{y}} \rangle_{\Lambda_M, T}^{\gamma_1, \gamma_2}$ is exponentially insensitive to boundary conditions and $\sum_{\gamma_1, \gamma_2} (-1)^{\delta_{\gamma_1} + \delta_{\gamma_2}} \Xi_{AT}^{\gamma_1, \gamma_2} / \Xi_{AT}$ is an $O(1)$ constant. Then from now on we will study only $\Xi_{AT}^-(\phi) \stackrel{def}{=} \widehat{\Xi}_{AT}^{(-, -), (-, -)}(\phi)$ and $\langle A_{\mathbf{x}} A_{\mathbf{y}} \rangle_{\Lambda_M, T}^{(-, -), (-, -)}$.

Proceeding as in Chapter 4 we integrate out the χ fields and we find:

$$\Xi_{AT}^-(\phi) = \int P_{Z_1, \sigma_1, \mu_1, C_1}(d\psi) e^{\mathcal{V}^{(1)} + \mathcal{B}^{(1)}}, \quad (8.5)$$

where

$$\mathcal{B}^{(1)}(\psi, \phi) = \sum_{m,n=1}^{\infty} \sum_{\substack{\mathbf{x}_1 \cdots \mathbf{x}_m \\ \mathbf{y}_1 \cdots \mathbf{y}_{2n}}}^{\underline{\sigma}, \underline{j}, \underline{\alpha}, \underline{\omega}} B_{m,2n;\underline{\sigma}, \underline{j}, \underline{\alpha}, \underline{\omega}}^{(1)}(\mathbf{x}_1, \dots, \mathbf{x}_m; \mathbf{y}_1, \dots, \mathbf{y}_{2n}) \left[\prod_{i=1}^m \phi_{\mathbf{x}_i} \right] \left[\prod_{i=1}^{2n} \partial_{j_i}^{\sigma_i} \psi_{\mathbf{y}_i, \omega_i}^{\alpha_i} \right]. \quad (8.6)$$

We proceed as for the partition function, namely as described in Chapter 5 above. We introduce the scale decomposition described in Chapter 5 and we perform iteratively the integration of the single scale fields, starting from the field of scale 1. After the integration of the fields $\psi^{(1)}, \dots, \psi^{(h+1)}$, $h_1^* < h \leq 0$, we are left with

$$\Xi_{AT}^-(\phi) = e^{-M^2 E_h + S^{(h+1)}(\phi)} \int P_{Z_h, \sigma_h, \mu_h, C_h}(d\psi^{\leq h}) e^{-\mathcal{V}^{(h)}(\sqrt{Z_h} \psi^{\leq h}) + \mathcal{B}^{(h)}(\sqrt{Z_h} \psi^{\leq h}, \phi)}, \quad (8.7)$$

where $P_{Z_h, \sigma_h, \mu_h, C_h}(d\psi^{\leq h})$ and $\mathcal{V}^{(h)}$ are the same as in Chapter 5, $S^{(h+1)}(\phi)$ denotes the sum of the contributions dependent on ϕ but independent of ψ , and finally $\mathcal{B}^{(h)}(\psi^{\leq h}, \phi)$ denotes the sum over all terms containing at least one ϕ field and two ψ fields. $S^{(h+1)}$ and $\mathcal{B}^{(h)}$ can be represented as

$$\begin{aligned} S^{(h+1)}(\phi) &= \sum_{m=1}^{\infty} \sum_{\mathbf{x}_1 \cdots \mathbf{x}_m} S_m^{(h+1)}(\mathbf{x}_1, \dots, \mathbf{x}_m) \prod_{i=1}^m \phi_{\mathbf{x}_i} \\ \mathcal{B}^{(h)}(\psi^{\leq h}, \phi) &= \sum_{m,n=1}^{\infty} \sum_{\substack{\mathbf{x}_1 \cdots \mathbf{x}_m \\ \mathbf{y}_1 \cdots \mathbf{y}_{2n}}}^{\underline{\sigma}, \underline{j}, \underline{\alpha}, \underline{\omega}} B_{m,2n;\underline{\sigma}, \underline{j}, \underline{\alpha}, \underline{\omega}}^{(h)}(\mathbf{x}_1, \dots, \mathbf{x}_m; \mathbf{y}_1, \dots, \mathbf{y}_{2n}) \left[\prod_{i=1}^m \phi_{\mathbf{x}_i} \right] \left[\prod_{i=1}^{2n} \partial_{j_i}^{\sigma_i} \psi_{\mathbf{y}_i, \omega_i}^{(\leq h) \alpha_i} \right]. \end{aligned} \quad (8.8)$$

Since the field ϕ is equivalent, as regarding dimensional bounds, to two ψ fields (see Theorem 8.1 below for a more precise statement), the only terms in the expansion for $\mathcal{B}^{(h)}$ which are not irrelevant are those with $m = n = 1$, $\sigma_1 = \sigma_2 = 0$ and they are marginal. Hence we extend the definition of the localization operator \mathcal{L} , so that its action on $\mathcal{B}^{(h)}(\psi^{\leq h}, \phi)$ is defined by its action on the kernels $\widehat{B}_{m,2n;\underline{\alpha}, \underline{\omega}}^{(h)}(\mathbf{q}_1, \dots, \mathbf{q}_m; \mathbf{k}_1, \dots, \mathbf{k}_{2n})$:

1) if $m = n = 1$ and $\alpha_1 + \alpha_2 = \omega_1 + \omega_2 = 0$, then $\mathcal{L} \widehat{B}_{1,2;\underline{\alpha}, \underline{\omega}}^{(h)}(\mathbf{q}_1; \mathbf{k}_1, \mathbf{k}_2) \stackrel{def}{=} \mathcal{P}_0 \widehat{B}_{1,2;\underline{\alpha}, \underline{\omega}}^{(h)}(\mathbf{k}_+; \mathbf{k}_+, \mathbf{k}_+)$, where \mathcal{P}_0 is defined as in (5.8);

2) in all other cases $\mathcal{L} \widehat{B}_{m,2n;\underline{\alpha}, \underline{\omega}}^{(h)} = 0$.

Using the symmetry considerations of §4.3 together with the remark that $\phi_{\mathbf{x}}$ is invariant under *Complex conjugation*, *Hole-particle* and (1) \longleftrightarrow (2), while under *Parity* $\phi_{\mathbf{x}} \rightarrow \phi_{-\mathbf{x}}$ and under *Rotation* $\phi_{(x, x_0)} \rightarrow \phi_{(-x_0, -x)}$, we easily realize that $\mathcal{L} \mathcal{B}^{(h)}$ has necessarily the following form:

$$\mathcal{L} \mathcal{B}^{(h)}(\psi^{\leq h}, \phi) = \frac{\overline{Z}_h}{Z_h} \sum_{\mathbf{x}, \omega} \frac{(-i\omega)}{2} \phi_{\mathbf{x}} \psi_{\omega, \mathbf{x}}^{(\leq h)+} \psi_{-\omega, \mathbf{x}}^{(\leq h)-}, \quad (8.9)$$

where \overline{Z}_h is real and $\overline{Z}_1 = a^{(1)}|_{\sigma=\mu=0} \equiv a^{(2)}|_{\sigma=\mu=0}$.

Note that apriori a term $\sum_{\mathbf{x}, \omega, \alpha} \phi_{\mathbf{x}} \psi_{\omega, \mathbf{x}}^{(\leq h) \alpha} \psi_{-\omega, \mathbf{x}}^{(\leq h) \alpha}$ is allowed by symmetry but, using (1) \longleftrightarrow (2) symmetry, one sees that its kernel is proportional to μ_k , $k \geq h$. So, with our definition of localization, such term contributes to $\mathcal{R} \mathcal{B}^{(h)}$.

Now that the action of \mathcal{L} on \mathcal{B} is defined, we can describe the single scale integration, for $h > h_1^*$. The integral in the r.h.s. of (8.7) can be rewritten as:

$$\begin{aligned} & e^{-M^2 t_h} \int P_{Z_{h-1}, \sigma_{h-1}, \mu_{h-1}, C_{h-1}}(d\psi^{\leq h-1}) \cdot \\ & \cdot \int P_{Z_{h-1}, \sigma_{h-1}, \mu_{h-1}, \tilde{f}_h^{-1}}(d\psi^{(h)}) e^{-\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}} \psi^{\leq h}) + \widehat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}} \psi^{\leq h}, \phi)}, \end{aligned} \quad (8.10)$$

where $\widehat{\mathcal{V}}^{(h)}$ was defined in (5.15) and

$$\widehat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}, \phi) \stackrel{\text{def}}{=} \mathcal{B}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)}, \phi). \quad (8.11)$$

Finally we define

$$\begin{aligned} e^{-\widetilde{E}_h M^2 + \widetilde{S}^{(h)}(\phi) - \mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leq h-1)}) + \mathcal{B}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leq h-1)}, \phi)} \stackrel{\text{def}}{=} \\ \stackrel{\text{def}}{=} \int P_{Z_{h-1}, \sigma_{h-1}, \mu_{h-1}, \widetilde{f}_h^{-1}}(d\psi^{(h)}) e^{-\widehat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}) + \widehat{\mathcal{B}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)}, \phi)}, \end{aligned} \quad (8.12)$$

and

$$E_{h-1} \stackrel{\text{def}}{=} E_h + t_h + \widetilde{E}_h, \quad S^{(h)}(\phi) \stackrel{\text{def}}{=} S^{(h+1)}(\phi) + \widetilde{S}^{(h)}(\phi). \quad (8.13)$$

With the definitions above, it is easy to verify that \overline{Z}_{h-1} satisfies the equation $\overline{Z}_{h-1} = \overline{Z}_h(1 + \overline{z}_h)$, where $\overline{z}_h = \overline{b}\lambda_h + O(\lambda^2)$, for some $\overline{b} \neq 0$. Then, for some $c > 0$, $\overline{Z}_1 e^{-c|\lambda|h} \leq \overline{Z}_h \leq \overline{Z}_1 e^{c|\lambda|h}$. The analogous of Theorem 5.1 for the kernels of $\mathcal{B}^{(h)}$ holds:

THEOREM 8.1. *Suppose that the hypothesis of Lemma 7.1 are satisfied. Then, for $h_1^* \leq \bar{h} \leq 1$ and a suitable constant C , the kernels of $\mathcal{B}^{(h)}$ satisfy*

$$\int d\mathbf{x}_1 \cdots d\mathbf{x}_{2n} |B_{2n, m; \underline{\sigma}, \underline{j}, \underline{\alpha}, \underline{\omega}}^{(\bar{h})}(\mathbf{x}_1, \dots, \mathbf{x}_m; \mathbf{y}_1, \dots, \mathbf{y}_{2n})| \leq M^2 \gamma^{-\bar{h}(D_k(n)+m)} (C|\lambda|)^{\max(1, n-1)}, \quad (8.14)$$

where $D_k(n) = -2 + n + k$ and $k = \sum_{i=1}^{2n} \sigma_i$.

Note that, consistently with our definition of localization, the dimension of $B_{2,1;(0,0),(+,-),(\omega,-\omega)}^{(h)}$ is $D_0(1) + 1 = 0$.

Again, proceeding as in Chapter 6, we can study the flow of \overline{Z}_h up to $h = -\infty$ and prove that $\overline{Z}_h = \overline{Z}_1 \gamma^{\overline{\eta}(h-1) + F_z^h}$, where $\overline{\eta}$ is a non trivial analytic function of λ (its linear part is non vanishing) and F_z^h is a suitable $O(\lambda)$ function (independent of σ_1, μ_1). We recall that $\overline{Z}_1 = O(1)$.

We proceed as above up to the scale h_1^* . Once that the scale h_1^* is reached we pass to the $\psi^{(1)}, \psi^{(2)}$ variables, we integrate out (say) the $\psi^{(1)}$ fields and we get

$$\int P_{Z_{h_1^*}, \widehat{m}_{h_1^*}^{(2)}, C_{h_1^*}}^{(2)}(d\psi^{(2)}(\leq h_1^*)) e^{-\overline{\mathcal{V}}^{(h_1^*)}(\sqrt{Z_{h_1^*}}\psi^{(2, \leq h_1^*)}) + \overline{\mathcal{B}}^{(h_1^*)}(\sqrt{Z_{h_1^*}}\psi^{(2, \leq h_1^*)})}, \quad (8.15)$$

with $\mathcal{L}\overline{\mathcal{B}}^{h_1^*}(\sqrt{Z_{h_1^*}}\psi^{(2, \leq h_1^*)}) = \overline{Z}_{h_1^*} \sum_{\mathbf{x}} i\phi_{\mathbf{x}} \psi_{1, \mathbf{x}}^{(2, \leq h_1^*)} \psi_{-1, \mathbf{x}}^{(2, \leq h_1^*)}$.

The scales $h_2^* \leq h \leq h_1^*$ are integrated as in Chapter 7 and one finds that the flow of \overline{Z}_h in this regime is trivial, i.e. if $h_2^* \leq h \leq h_1^*$, $\overline{Z}_h = \overline{Z}_{h_1^*} \gamma^{F_z^h}$, with $F_z^h = O(\lambda)$.

The result is that the correlation function $\langle H_{\mathbf{x}}^{AT} H_{\mathbf{y}}^{AT} \rangle_{\Lambda_M, T}$ is given by a convergent power series in λ , uniformly in Λ_M . Then, the leading behaviour of the specific heat is given by the sum over \mathbf{x} and \mathbf{y} of the lowest order contributions to $\langle H_{\mathbf{x}}^{AT} H_{\mathbf{y}}^{AT} \rangle_{\Lambda_M, T}$, namely by the diagrams in Fig.6. Absolute convergence of the power series of $\langle H_{\mathbf{x}}^{AT} H_{\mathbf{y}}^{AT} \rangle_{\Lambda_M, T}$ implies that the rest is a small correction.

The conclusion is that C_v , for λ small and $|t - \sqrt{2} + 1|, |u| \leq (\sqrt{2} - 1)/4$, is given by:

$$\begin{aligned} C_v = \frac{\beta^2}{|\Lambda|} \sum_{\mathbf{x}, \mathbf{y} \in \Lambda_M} \sum_{\omega_1, \omega_2 = \pm 1} \sum_{h, h' = h_2^*}^1 \frac{(Z_{h \vee h'}^{(1)})^2}{Z_{h-1} Z_{h'-1}} \left[G_{(+, \omega_1), (+, \omega_2)}^{(h)}(\mathbf{x} - \mathbf{y}) G_{(-, -\omega_2), (-, -\omega_1)}^{(h')}(\mathbf{y} - \mathbf{x}) + \right. \\ \left. + G_{(+, \omega_1), (-, -\omega_2)}^{(h)}(\mathbf{x} - \mathbf{y}) G_{(-, -\omega_1), (+, \omega_2)}^{(h')}(\mathbf{x} - \mathbf{y}) \right] + \frac{1}{|\Lambda|} \sum_{\mathbf{x}, \mathbf{y} \in \Lambda_M} \sum_{h_2^*}^1 \left(\frac{\overline{Z}_h}{Z_h} \right)^2 \Omega_{\Lambda_M}^{(h)}(\mathbf{x} - \mathbf{y}), \end{aligned} \quad (8.16)$$

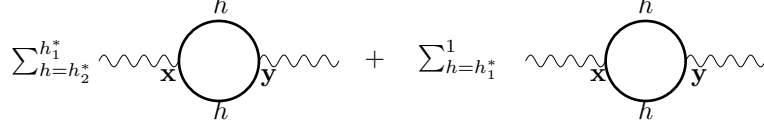


FIG. 6. The lowest order diagrams contributing to $\langle H_{\mathbf{x}}^{AT} H_{\mathbf{y}}^{AT} \rangle_{\Lambda_M, T}$. The wavy lines ending in the points labeled \mathbf{x} and \mathbf{y} represent the fields $\phi_{\mathbf{x}}$ and $\phi_{\mathbf{y}}$ respectively. The solid lines labeled by h and going from \mathbf{x} to \mathbf{y} represent the propagators $g^{(h)}(\mathbf{x}-\mathbf{y})$. The sums are over the scale indices and, even if not explicitly written, over the indexes $\underline{\alpha}, \underline{\omega}$ (and the propagators depend on these indexes).

where $h \vee h' = \max\{h, h'\}$ and $G_{(\alpha_1, \omega_1), (\alpha_2, \omega_2)}^{(h)}(\mathbf{x})$ must be interpreted as

$$G_{(\alpha_1, \omega_1), (\alpha_2, \omega_2)}^{(h)}(\mathbf{x}) = \begin{cases} g_{(\alpha_1, \omega_1), (\alpha_2, \omega_2)}^{(h)}(\mathbf{x}) & \text{if } h > h_1^*, \\ g_{\omega_1, \omega_2}^{(1, \leq h_1^*)}(\mathbf{x}) + g_{\omega_1, \omega_2}^{(2, h_1^*)}(\mathbf{x}) & \text{if } h = h_1^*, \\ g_{\omega_1, \omega_2}^{(2, h)}(\mathbf{x}) & \text{if } h_2^* < h < h_1^*, \\ g_{\omega_1, \omega_2}^{(2, \leq h_2^*)}(\mathbf{x}) & \text{if } h = h_2^*. \end{cases}$$

Moreover, if $N, n_0, n_1 \geq 0$ and $n = n_0 + n_1$, $|\partial_x^{n_0} \partial_{x_0} \Omega_{\Lambda_M}^{(h)}(\mathbf{x})| \leq C_{N, n} |\lambda| \frac{\gamma^{(2+n)h}}{1 + (\gamma^h |\mathbf{d}(\mathbf{x})|)^N}$. Now, calling η_c the exponent associated to \bar{Z}_h/Z_h , from (8.16) we find:

$$C_v = -C_1 \gamma^{2\eta_c h_1^*} \log_\gamma \gamma^{h_1^* - h_2^*} (1 + \Omega_{h_1^*, h_2^*}^{(1)}(\lambda)) + C_2 \frac{1 - \gamma^{2\eta_c (h_1^* - 1)}}{2\eta_c} (1 + \Omega_{h_1^*}^{(2)}(\lambda)), \quad (8.17)$$

where $|\Omega_{h_1^*, h_2^*}^{(1)}(\lambda)|, |\Omega_{h_1^*}^{(2)}(\lambda)| \leq c|\lambda|$, for some c , and C_1, C_2 are functions of λ, t, u , bounded above and below by $O(1)$ constants. Note that, defining Δ as in the line following (1.8), $\gamma^{(1-\eta_\sigma)h_1^*} \Delta^{-1}$ is bounded above and below by $O(1)$ constants. Then, using (7.30), (1.8) follows.

9. Conclusions and open problems.

In the previous Chapters we described a constructive approach to the study of the thermodynamic properties of weakly interacting spin systems in two dimensions, arbitrarily near to the critical temperature(s).

The approach was based on an exact mapping of the interacting spin system into an interacting system of $1 + 1$ -dim non relativistic fermions. It applies to a wide class of perturbations of Ising, including the Ashkin–Teller model, the 8 vertex model, the next to nearest neighbor Ising model and linear combinations of these models.

As an application of the method, we studied the free energy and the specific heat for the anisotropic Ashkin–Teller model, which is a well-known model of statistical mechanics, widely studied with a number of different theoretical and empirical techniques. However exact results were lacking since long time: in the 1970’s Baxter, Kadanoff and others conjectured that (1) anisotropic AT has in general two different critical temperatures (whose location was unknown) and (2) AT belongs to the same universality class of Ising except at the isotropic point.

Our calculation of the free energy and of the specific heat allowed to (rigorously) prove for the first time the two conjectures above in the regime of weak coupling and to derive detailed asymptotic expressions for the specific heat itself and for the shape of the critical surface (*i.e.* for the critical temperatures as functions of the anisotropy parameter and of the coupling). The latter calculation revealed the existence of a previously unknown critical exponent, describing how the difference of the critical temperatures rescale, when we let the anisotropy go to 0.

Important open problems are the following.

1) The study of the free energy and of the correlation functions *directly at the critical point*, where it is expected that the correlation functions are, in the thermodynamic limit, homogeneous functions of the coordinates and, moreover, *conformal invariant*. Even for the Ising model this is a widely expected by still unproved conjecture. Technically we have to face the difficulty of dealing with a linear combination of 16 Grassmann partition functions, differing for the boundary conditions; up to now we are able to control this combination only outside the critical point (but arbitrarily near to it).

2) The study of more complicated correlation functions, such as the spin–spin correlation functions $\langle \sigma_{\mathbf{0}} \sigma_{\mathbf{x}} \rangle$. These are difficult to study in the Grassmann formulation, because they correspond to the average of an exponential of a relevant non traslationally invariant operator in the Grassmann fields. Such operator is concentrated along a path connecting the two points $\mathbf{0}, \mathbf{x}$ and, moreover, is weighted by an order 1 constant! Note that even in the free case (Ising) the calculation of the spin–spin correlation functions is very non trivial and is based on the analysis of a Toeplitz determinant, in the limit in which the size of the Toeplitz matrix diverging to infinity, through an application of Szego’s Theorem.

A first step towards the understanding of such objects would be the calculation of averages of exponentials of simpler relevant non translational invariant operators, such as those appearing in the study of large deviations for the magnetization or the particle number in a bounded region of \mathbb{Z}^2 .

Appendix A1. Grassmann integration. Truncated expectations.

In the present Appendix we list some more properties of Grassmann integration (the basic ones were introduced in §2.2). In particular we introduce the definition of truncated expectation, and we describe a possible graphical interpretation for the truncated expectations, the so-called *Feynman diagrams*.

A1.1. Truncated expectations and some more rules.

Pursuing further the analogy with Gaussian integrals stressed in §2.2, we can consider a “measure” (a similar expression is found replacing g with a matrix, see (A1.17) below)

$$P(d\psi) = \prod_{\alpha \in A} d\psi_{\alpha}^{+} d\psi_{\alpha}^{-} g_{\alpha} e^{-\sum_{\alpha \in A} \psi_{\alpha}^{+} g_{\alpha}^{-1} \psi_{\alpha}^{-}} ; \quad (A1.1)$$

by construction one has

$$\int P(d\psi) = 1 , \quad \int P(d\psi) \psi_{\alpha}^{-} \psi_{\beta}^{+} = \delta_{\alpha, \beta} g_{\alpha} . \quad (A1.2)$$

In general $P(d\psi)$ will be called a *Gaussian fermionic integration measure* (or *Grassman integration measure* or, as we shall do in the following, integration *tout court*) with covariance g : for any analytic function F defined on the Grassman algebra we can write

$$\int P(d\psi) F(\psi) = \mathcal{E}(F) . \quad (A1.3)$$

However note that $P(d\psi)$ is not at all a real measure, as it does not satisfy the necessary positivity conditions, so that the terminology is only formal and the use of the symbol \mathcal{E} (which stands for expectation value) is meant only by analogy.

Given p functions X_1, \dots, X_p defined on the Grassman algebra and p positive integer numbers n_1, \dots, n_p , the *truncated expectation* is defined as

$$\mathcal{E}^T(X_1, \dots, X_p; n_1, \dots, n_p) = \frac{\partial^{n_1 + \dots + n_p}}{\partial \lambda_1^{n_1} \dots \partial \lambda_p^{n_p}} \log \int P(d\psi) e^{\lambda_1 X_1(\psi) + \dots + \lambda_p X_p(\psi)} \Big|_{\lambda=0} , \quad (A1.4)$$

where $\lambda = \{\lambda_1, \dots, \lambda_p\}$. It is easy to check that \mathcal{E}^T is a linear operation, that is, formally,

$$\mathcal{E}^T(c_1 X_1 + \dots + c_p X_p; n) = \sum_{n_1 + \dots + n_p = n} \frac{n!}{n_1! \dots n_p!} c_1^{n_1} \dots c_p^{n_p} \mathcal{E}^T(X_1, \dots, X_p; n_1, \dots, n_p) , \quad (A1.5)$$

so that the following relations immediately follow:

$$\begin{aligned} (1) \quad & \mathcal{E}^T(X; 1) = \mathcal{E}(X) , \\ (2) \quad & \mathcal{E}^T(X; 0) = 0 , \\ (3) \quad & \mathcal{E}^T(X, \dots, X; n_1, \dots, n_p) = \mathcal{E}^T(X; n_1 + \dots + n_p) . \end{aligned} \quad (A1.6)$$

Moreover one has

$$\mathcal{E}^T(X_1, \dots, X_1, \dots, X_p, \dots, X_p; 1, \dots, 1, \dots, 1, \dots, 1) = \mathcal{E}^T(X_1, \dots, X_p; n_1, \dots, n_p) , \quad (A1.7)$$

where, for any $j = 1, \dots, p$, in the l.h.s. the function X_j is repeated n_j times and 1 is repeated $n_1 + \dots + n_p$ times.

We define also

$$\mathcal{E}^T(X_1, \dots, X_p) \equiv \mathcal{E}^T(X_1, \dots, X_p; 1, \dots, 1). \quad (\text{A1.8})$$

By (A1.7) we see that all truncated expectations can be expressed in terms of (A1.8); it is easy to see that (A1.8) is vanishing if $X_j = 0$ for at least one j .

The truncated expectation appears naturally considering the integration of an exponential; in fact as a particular case of (A1.4) one has

$$\mathcal{E}^T(X; n) = \frac{\partial^n}{\partial \lambda^n} \log \int P(d\psi) e^{\lambda X(\psi)} \Big|_{\lambda=0}, \quad (\text{A1.9})$$

so that

$$\begin{aligned} \log \int P(d\psi) e^{X(\psi)} &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \log \sum_{n=0}^{\infty} \int P(d\psi) e^{\lambda X(\psi)} \Big|_{\lambda=0} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{E}^T(X; n). \end{aligned} \quad (\text{A1.10})$$

The following properties, immediate consequence of (2.6) and very similar to the properties of Gaussian integrations, follow.

(1) *Wick rule.* Given two sets of labels $\{\alpha_1, \dots, \alpha_n\}$ and $\{\beta_1, \dots, \beta_m\}$ in A , one has

$$\int P(d\psi) \psi_{\alpha_1}^- \dots \psi_{\alpha_n}^- \psi_{\beta_1}^+ \dots \psi_{\beta_m}^+ = \delta_{n,m} \sum_{\pi} (-1)^{p_{\pi}} \prod_{i=1}^n \delta_{\alpha_i, \beta_{\pi(i)}} g_{\alpha_i}, \quad (\text{A1.11})$$

where the sum is over all the permutations $\pi = \{\pi(1), \dots, \pi(n)\}$ of the indices $\{1, \dots, n\}$ with parity p_{π} with respect to the fundamental permutation.

(2) *Addition principle.* Given two integrations $P(d\psi_1)$ and $P(d\psi_2)$, with covariance g_1 and g_2 respectively, then, for any function F which can be written as sum over monomials of Grassman variables, *i.e.* $F = F(\psi)$, with $\psi = \psi_1 + \psi_2$, one has

$$\int P(d\psi_1) \int P(d\psi_2) F(\psi_1 + \psi_2) = \int P(d\psi) F(\psi), \quad (\text{A1.12})$$

where $P(d\psi)$ has covariance $g \equiv g_1 + g_2$. It is sufficient to prove it for $F(\psi) = \psi^- \psi^+$, then one uses the anticommutation rules (2.5). One has

$$\begin{aligned} &\int P(d\psi_1) \int P(d\psi_2) (\psi_1^- + \psi_2^-) (\psi_1^+ + \psi_2^+) \\ &= \int P(d\psi_1) \psi_1^- \psi_1^+ \int P(d\psi_2) + \int P(d\psi_1) \int P(d\psi_2) \psi_2^- \psi_2^+ = g_1 + g_2. \end{aligned} \quad (\text{A1.13})$$

where (A1.2) has been used.

(3) *Invariance of exponentials.* From the definition of truncated expectations, it follows that, if ϕ is an “external field”, *i.e.* a not integrated field, then

$$\int P(d\psi) e^{X(\psi+\phi)} = \exp \left[\sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{E}^T(X(\cdot + \phi); n) \right] \equiv e^{X'(\phi)}, \quad (\text{A1.14})$$

which is a main technical point: (A1.14) says that integrating an exponential one still gets an exponential, whose argument is expressed by the sum of truncated expectations.

(4) *Change of integration.* If $P_g(d\psi)$ denotes the integration with covariance g , then, for any analytic function $F(\psi)$, one has

$$\frac{1}{\mathcal{N}_\nu} \int P_g(d\psi) e^{-\nu\psi^+\psi^-} F(\psi) = \int P_{\tilde{g}}(d\psi) F(\psi) , \quad \tilde{g}^{-1} = g^{-1} + \nu , \quad (\text{A1.15})$$

where

$$\mathcal{N}_\nu = \frac{g^{-1} + \nu}{g^{-1}} = 1 + g\nu = \int P_g(d\psi) e^{-\nu\psi^+\psi^-} . \quad (\text{A1.16})$$

The proof is very easy from the definitions. More generally one has that, if A is a set of labels for the Grassmann fields, if M is an invertible $|A| \times |A|$ matrix and $P_M(d\psi)$ is given by

$$P_M(d\psi) = \int \left(\prod_{\alpha \in A} d\psi_\alpha^+ d\psi_\alpha^- \right) \det M e^{-\sum_{i,j \in A} \psi_i^+ M_{ij}^{-1} \psi_j^-} , \quad (\text{A1.17})$$

then

$$\frac{1}{\mathcal{N}_N} \int P_M(d\psi) e^{-\sum_{i,j \in A} \psi_i^+ N_{ij}^{-1} \psi_j^-} F(\psi) = \int P_{\tilde{M}}(d\psi) F(\psi) , \quad (\text{A1.18})$$

where

$$\tilde{M}^{-1} = M^{-1} + N^{-1} \quad (\text{A1.19})$$

and

$$\mathcal{N}_N = \det(\mathbb{1} + N^{-1}M) = \frac{\det(M^{-1} + N^{-1})}{\det M^{-1}} = \int P_M(d\psi) e^{-\sum_{i,j \in A} \psi_i^+ N_{ij}^{-1} \psi_j^-} . \quad (\text{A1.20})$$

A1.2. Graphical representation for truncated expectations.

Given a Grassman algebra as in (2.5) and an integration measure like (A1.1) we define the *simple expectation* as in (A1.3). Then

$$g_\alpha = \mathcal{E}(\psi_\alpha^- \psi_\alpha^+) . \quad (\text{A1.21})$$

Given a monomial

$$X(\psi) \equiv \tilde{\psi}_B = \prod_{\alpha \in B} \psi_\alpha^{\sigma_\alpha} , \quad (\text{A1.22})$$

where B is a subset of A and $\sigma_\alpha \in \{\pm\}$, the expectation $\mathcal{E}(\tilde{\psi}_B)$ can be graphically represented in the following way.

Represent the indices $\alpha \in B$ as points on the plane. With each ψ_α^+ , $\alpha \in B$, we associate a line exiting from α , while with each ψ_α^- , $\alpha \in B$, we associate a line entering α . Let \mathcal{T} be the set of graphs obtained by contracting such lines in all possible ways so that only lines with opposite σ_α are contracted: given $\alpha, \beta \in B$, denote by $(\alpha\beta)$ the line joining α and β and by τ an element of \mathcal{T} , *i.e.* a graph in \mathcal{T} .

Then we can easily verify that

$$\mathcal{E}(\tilde{\psi}_B) = \sum_{\tau \in \mathcal{T}} \prod_{(\alpha\beta) \in \tau} (-1)^{\pi_\tau} g_\alpha \delta_{\alpha,\beta} , \quad (\text{A1.23})$$

which is the *Wick rule* stated in §A1.1: here π_τ is a sign which depends on the graph τ (see (A1.11)).

Then define the *truncated expectation*

$$\mathcal{E}^T(\tilde{\psi}_{B_1}, \dots, \tilde{\psi}_{B_p}; n_1, \dots, n_p) , \quad (\text{A1.24})$$

with $B_j \subset A$ for any j , as in (A1.4).

One easily check that, if X_j are analytic functions of the Grassman variables (each depending on an even number of variables, for simplicity, so that no change of sign intervenes in permuting the order of the X_j), then

$$\begin{aligned}
(1) \quad & \mathcal{E}^T(X_1, X_2) = \mathcal{E}(X_1 X_2) - \mathcal{E}(X_1) \mathcal{E}(X_2) = \mathcal{E}(X_1 X_2) - \mathcal{E}^T(X_1) \mathcal{E}^T(X_2) , \\
(2) \quad & \mathcal{E}^T(X_1, X_2, X_3) = \mathcal{E}(X_1 X_2 X_3) - \mathcal{E}(X_1 X_2) \mathcal{E}(X_3) - \mathcal{E}(X_1 X_3) \mathcal{E}(X_2) \\
& \quad - \mathcal{E}(X_2 X_3) \mathcal{E}(X_1) + 2\mathcal{E}(X_1) \mathcal{E}(X_2) \mathcal{E}(X_3) = \mathcal{E}(X_1 X_2 X_3) - \\
& \quad - \mathcal{E}^T(X_1 X_2) \mathcal{E}^T(X_3) - \mathcal{E}^T(X_1 X_3) \mathcal{E}^T(X_2) - \mathcal{E}^T(X_2 X_3) \mathcal{E}^T(X_1) , \\
(3) \quad & \mathcal{E}^T(X_1, X_2, X_3, X_4) = \mathcal{E}(X_1 X_2 X_3 X_4) - \mathcal{E}(X_1 X_2 X_3) \mathcal{E}(X_4) - \mathcal{E}(X_1 X_2 X_4) \mathcal{E}(X_3) \\
& \quad - \mathcal{E}(X_1 X_3 X_4) \mathcal{E}(X_2) - \mathcal{E}(X_2 X_3 X_4) \mathcal{E}(X_1) \\
& \quad - \mathcal{E}(X_1 X_2) \mathcal{E}(X_3 X_4) - \mathcal{E}(X_1 X_3) \mathcal{E}(X_2 X_4) - \mathcal{E}(X_1 X_4) \mathcal{E}(X_2 X_3) \\
& \quad + 2\mathcal{E}(X_1 X_2) \mathcal{E}(X_3) \mathcal{E}(X_4) + 2\mathcal{E}(X_1 X_3) \mathcal{E}(X_2) \mathcal{E}(X_4) + 2\mathcal{E}(X_1 X_4) \mathcal{E}(X_2) \mathcal{E}(X_3) \\
& \quad + 2\mathcal{E}(X_2 X_3) \mathcal{E}(X_1) \mathcal{E}(X_4) + 2\mathcal{E}(X_2 X_4) \mathcal{E}(X_1) \mathcal{E}(X_3) + 2\mathcal{E}(X_3 X_4) \mathcal{E}(X_1) \mathcal{E}(X_2) \\
& \quad - 6\mathcal{E}(X_1) \mathcal{E}(X_2) \mathcal{E}(X_3) \mathcal{E}(X_4) = \mathcal{E}(X_1 X_2 X_3 X_4) - \mathcal{E}^T(X_1 X_2 X_3) \mathcal{E}^T(X_4) - \\
& \quad - \mathcal{E}^T(X_1 X_2 X_4) \mathcal{E}^T(X_3) - \mathcal{E}^T(X_1 X_3 X_4) \mathcal{E}^T(X_2) - \mathcal{E}^T(X_2 X_3 X_4) \mathcal{E}^T(X_1) - \\
& \quad - \mathcal{E}^T(X_1 X_2) \mathcal{E}^T(X_3 X_4) - \mathcal{E}^T(X_1 X_3) \mathcal{E}^T(X_2 X_4) - \mathcal{E}^T(X_1 X_4) \mathcal{E}^T(X_2 X_3) .
\end{aligned} \tag{A1.25}$$

and so on. One can always write the truncated expectations in terms of simple expectations and viceversa: it is easy to check that in general one has

$$\mathcal{E}(X_1 \dots X_s) = \sum_{p=1}^s \sum_{Y_1, \dots, Y_p} \mathcal{E}^T(X_{\pi_1(1)}, \dots, X_{\pi_{|Y_1|}(1)}) \dots \mathcal{E}^T(X_{\pi_1(p)}, \dots, X_{\pi_{|Y_p|}(p)}) , \tag{A1.26}$$

where:

- (1) the sum is over all the possible sets Y_i , $i = 1, \dots, p$, which are unions of $|Y_i|$ sets X_j , such that $\cup_{j=1}^s X_j = \cup_{k=1}^{|Y_1|+\dots+|Y_p|} Y_k$;
 - (2) $\{\pi_1(1), \dots, \pi_{|Y_1|}(1), \pi_1(2), \dots, \dots, \pi_{|Y_p|}(p)\}$ is a permutation of $\{1, \dots, s\}$.
- (A1.26) can be verified by induction, see Appendix A4 in [G].

We can now describe the rules to graphically represent the truncated expectations $\mathcal{E}^T(\tilde{\psi}_{B_1}, \dots, \tilde{\psi}_{B_p})$ in (A1.24). Draw in the plane p boxes G_1, \dots, G_p , such that G_i contains all points representing the indices belonging to B_i ; from each of the points $\alpha \in G_i$ draw the line corresponding to the field $\psi_{\alpha}^{\sigma_{\alpha}}$ contained in the monomial $\tilde{\psi}_{B_i}$, with the direction consistent with σ_{α} (the line enters or exists α depending if σ_{α} is $-$ or $+$). We call *clusters* such boxes. Then consider all possible graphs τ obtained by contracting as before all the lines emerging from the points in such a way that no line is left uncontracted and with the property that if the clusters were considered as points then τ would be connected. If we denote the lines as before we have

$$\mathcal{E}^T(\tilde{\psi}_{B_1}, \dots, \tilde{\psi}_{B_p}; n_1, \dots, n_p) = \sum_{\tau \in \mathcal{T}_0} \prod_{(\alpha\beta) \in \tau} (-1)^{\pi_{\tau}} g_{\alpha} \delta_{\alpha, \beta} , \tag{A1.27}$$

where \mathcal{T}_0 denotes the set of all graphs obtained following the just given prescription; again π_{τ} is a sign depending on τ .

The reason why we have to sum only over the connected graphs follows from (A1.26), as it can be easily verified by induction.

Appendix A2. The Pfaffian expansion.

In this Appendix we prove (4.14).

Given s set of indices P_1, \dots, P_s , consider the quantity $\mathcal{E}^T(\tilde{\phi}(P_1), \dots, \tilde{\phi}(P_s))$, with $\tilde{\phi}(P_i) = \prod_{f \in P_i} \phi_{\mathbf{x}(f), \omega(f)}^{\alpha(f)}$ and $\phi = \chi, \psi$.

Define

$$\mathcal{D}\phi = \prod_{j=1}^n \prod_{f \in P_j} d\phi_{\mathbf{x}(f), \omega(f)}^{\alpha(f)} \quad (\phi, G\phi) = \sum_{f, f' \in \cup_i P_i} \phi_{\mathbf{x}(f), \omega(f)}^{\alpha(f)} G_{f, f'} \phi_{\mathbf{x}(f'), \omega(f')}^{\alpha(f')} \quad (\text{A2.1})$$

where, if $2n = \sum_{j=1}^s |P_j|$ then G is the $2n \times 2n$ antisymmetric matrix with entries

$$G_{f, f'} \stackrel{\text{def}}{=} \langle \phi_{\mathbf{x}(f), \omega(f)}^{\alpha(f)} \phi_{\mathbf{x}(f'), \omega(f')}^{\alpha(f')} \rangle . \quad (\text{A2.2})$$

Then one has

$$\mathcal{E} \left(\prod_{j=1}^s \tilde{\phi}(P_j) \right) = \text{Pf } G = \int \mathcal{D}\phi \exp \left[-\frac{1}{2} (\phi, G\phi) \right] . \quad (\text{A2.3})$$

Setting $X \equiv \{1, \dots, s\}$ and

$$\bar{V}_{jj'} = \frac{1}{2} \sum_{f \in P_j} \sum_{f' \in P_{j'}} \phi_{\mathbf{x}(f), \omega(f)}^{\alpha(f)} G_{f, f'} \phi_{\mathbf{x}(f'), \omega(f')}^{\alpha(f')} , \quad (\text{A2.4})$$

we write

$$V(X) = \sum_{j, j' \in X} \bar{V}_{jj'} = \sum_{j \leq j'} V_{jj'} , \quad (\text{A2.5})$$

so defining the quantity $V_{jj'}$ as

$$V_{jj'} = \begin{cases} \bar{V}_{jj'} , & \text{if } j = j' , \\ \bar{V}_{jj'} + \bar{V}_{j'j} , & \text{if } j < j' . \end{cases} \quad (\text{A2.6})$$

Then (A2.3) can be written, by the definition of Grassman variables, as

$$\mathcal{E} \left(\prod_{j=1}^s \tilde{\phi}(P_j) \right) = \int \mathcal{D}\phi e^{-V(X)} . \quad (\text{A2.7})$$

We now want to express the last expression in terms of the functions W_X , defined as follows:

$$W_X(X_1, \dots, X_r; t_1, \dots, t_r) = \sum_{\ell} \prod_{k=1}^r t_k(\ell) V_{\ell} , \quad (\text{A2.8})$$

where:

(1) X_k are subsets of X with $|X_k| = k$, inductively defined as:

$$\begin{cases} X_1 = \{1\} , \\ X_{k+1} \supset X_k , \end{cases} \quad (\text{A2.9})$$

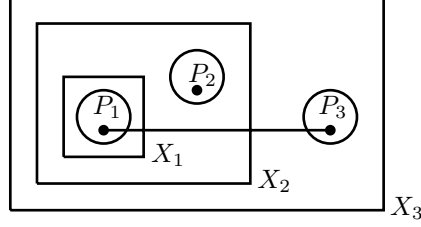


FIG. A2.1. Graphical representation of the sets X_k , $k=1,2,3$. In the example $X_1=\{1\}$, $X_2=\{1,2\}$ and $X_3=\{1,2,3\}$. The $\ell=(13)$ intersects the boundaries of X_1 and of X_2 .

- (2) $\ell = (jj')$ is a pair of elements $j, j' \in X$ and the sum in (A2.8) is over all the possible pairs (jj') ,
 (3) the functions $t_k(\ell)$ are defined as follows:

$$t_k(\ell) = \begin{cases} t_k, & \text{if } \ell \sim \partial X_k, \\ 1, & \text{otherwise,} \end{cases} \quad (\text{A2.10})$$

where $\ell \sim X_k$ means that $\ell = (jj')$ “intersects the boundary” of X_k , *i.e.* connects a point in P_j , $j \in X_k$, to a point in $P_{j'}$, $j' \notin X_k$. See Fig. A2.1.

From definition (A2.8) it follows:

$$W_X(X_1; t_1) = \sum_{j=2}^s t_1 V_{1j} + V_{11} + \sum_{1 < j' \leq j} V_{j'j} = (1 - t_1) [V(X_1) + V(X \setminus X_1)] + t_1 V(X) \quad (\text{A2.11})$$

so that

$$\begin{aligned} e^{-V(X)} &= \int_0^1 dt_1 \left[\frac{\partial}{\partial t_1} e^{-W_X(X_1; t_1)} \right] + e^{-W_X(X_1; 0)} \\ &= - \sum_{\ell_1 \sim \partial X_1} V_{\ell_1} \int_0^1 dt_1 e^{-W_X(X_1; t_1)} + e^{-W_X(X_1; 0)}. \end{aligned} \quad (\text{A2.12})$$

Again by definition we have:

$$\begin{aligned} W_X(X_1, X_2; t_1, t_2) &= \\ &= V_{11} + t_1 V_{12} + t_1 t_2 \sum_{j=3}^s V_{1j} + V_{22} + t_2 \sum_{j=3}^s V_{2j} + \sum_{2 < j' \leq j} V_{j'j} = \\ &= t_1 t_2 \sum_{j=2}^s V_{1j} + t_2 V_{11} + t_2 \sum_{1 < j' \leq j} V_{j'j} + (1 - t_2) \left[V_{11} + t_1 V_{12} + V_{22} + \sum_{2 < j' \leq j} V_{j'j} \right] = \\ &= t_2 W_X(X_1; t_1) + (1 - t_2) [W_{X_2}(X_1; t_1) + V(X \setminus X_2)] \end{aligned} \quad (\text{A2.13})$$

If we define $X_2 \equiv X_1 \cup \ell_1$, *i.e.* $X_2 = \{1, \text{point connected by } \ell_1 \text{ with } 1\}$, then:

$$\begin{aligned} e^{-W_X(X_1; t_1)} &= \int_0^1 dt_2 \left[\frac{\partial}{\partial t_2} e^{-W_X(X_1, X_2; t_1, t_2)} \right] + e^{-W_X(X_1, X_2; t_1, 0)} \\ &= - \sum_{\ell_2 \sim \partial X_2} V_{\ell_2} \int_0^1 dt_2 t_1(\ell_2) e^{-W_X(X_1, X_2; t_1, t_2)} + e^{-W_X(X_1, X_2; t_1, 0)}. \end{aligned} \quad (\text{A2.14})$$

Substituting (A2.14) in (A2.12) we get:

$$\begin{aligned}
e^{-V(X)} &= \sum_{\ell_1 \sim \partial X_1} \sum_{\ell_2 \sim \partial X_2} \int_0^1 dt_1 \int_0^1 dt_2 (-1)^2 V_{\ell_1} V_{\ell_2} t_1(\ell_2) e^{-W_X(X_1, X_2; t_1, t_2)} \\
&+ \sum_{\ell_1 \sim \partial X_1} \int_0^1 dt_1 (-1) V_{\ell_1} e^{-W_X(X_1, X_2; t_1, 0)} + e^{-W_X(X_1; 0)} .
\end{aligned} \tag{A2.15}$$

A relation generalizing (A2.13) holds:

$$\begin{aligned}
W_X(X_1, \dots, X_{p+1}; t_1, \dots, t_{p+1}) &= t_{p+1} W_X(X_1, \dots, X_p; t_1, \dots, t_p) + \\
(1 - t_{p+1}) [W_{X_{p+1}}(X_1, \dots, X_p; t_1, \dots, t_p) &+ V(X \setminus X_{p+1})]
\end{aligned} \tag{A2.16}$$

where $p < s$. In fact in the sum over ℓ in (A2.8) we can distinguish two cases: either $\ell \sim X_{p+1}$ or $\ell \not\sim X_{p+1}$. In the former case V_ℓ is necessarily multiplied by t_{p+1} and, if $\ell = (j'j)$, $j' \leq p+1$, $j > p+1$; in the latter case V_ℓ is not multiplied by t_{p+1} and either $j', j \leq p+1$ or $j', j > p+1$. Then, clearly:

$$\begin{aligned}
W_X(X_1, \dots, X_{p+1}; t_1, \dots, t_{p+1}) &= \\
&= t_{p+1} [W_X(X_1, \dots, X_p; t_1, \dots, t_p) - W_{X_{p+1}}(X_1, \dots, X_p; t_1, \dots, t_p) - W_{X \setminus X_{p+1}}(X_1, \dots, X_p; t_1, \dots, t_p)] + \\
&+ W_{X_{p+1}}(X_1, \dots, X_p; t_1, \dots, t_p) + W_{X \setminus X_{p+1}}(X_1, \dots, X_p; t_1, \dots, t_p)
\end{aligned} \tag{A2.17}$$

that is equivalent to (A2.16). We can iterate the procedure followed to get (A2.12) and (A2.15). In the general case we find:

$$\begin{aligned}
e^{-V(X)} &= \sum_{r=0}^{s-1} \sum_{\ell_1 \sim \partial X_1} \dots \sum_{\ell_r \sim \partial X_r} \int_0^1 dt_1 \dots \int_0^1 dt_r (-1)^r V_{\ell_1} \dots V_{\ell_r} \\
&\left(\prod_{k=1}^{r-1} t_1(\ell_{k+1}) \dots t_k(\ell_{k+1}) \right) e^{-W_X(X_1, \dots, X_{r+1}; t_1, \dots, t_r, 0)} ,
\end{aligned} \tag{A2.18}$$

where the meaningless factors must be replaced by 1. Moreover, from (A2.16) we soon realize that

$$\begin{aligned}
W_X(X_1, \dots, X_s; t_1, \dots, t_{s-1}, 0) &= W_X(X_1, \dots, X_{s-1}; t_1, \dots, t_{s-1}) \\
W_X(X_1, \dots, X_r; t_1, \dots, t_{r-1}, 0) &= W_{X_r}(X_1, \dots, X_{r-1}; t_1, \dots, t_{r-1}) + V(X \setminus X_r)
\end{aligned} \tag{A2.19}$$

The last equation holds for $r > 1$. If $r = 1$:

$$W_X(X_1; 0) = V(X_1) + V(X \setminus X_1) \tag{A2.20}$$

Let T be a tree graph connecting X_1, \dots, X_r , such that:

- (1) for all $k = 1, \dots, r$, T is “anchored” to some point (j, i) , i.e. T contains a line incident with (j, i) , where $j \in X_k$ and $i \in \{1, \dots, |P_j^\pm|\}$,
- (2) each line $\ell \in T$ intersects at least one boundary ∂X_k ,
- (3) the lines ℓ_1, ℓ_2, \dots are ordered in such a way that $\ell_1 \sim \partial X_1, \ell_2 \sim \partial X_2, \dots$,
- (4) for each $\ell \in T$ there exist two indexes $n(\ell)$ and $n'(\ell)$ defined as follows:

$$\begin{aligned}
n(\ell) &= \max\{k : \ell \sim \partial X_k\} , \\
n'(\ell) &= \min\{k : \ell \sim \partial X_k\} .
\end{aligned} \tag{A2.21}$$

We shall say that T is an *anchored tree*.

Using the definitions above, we can rewrite (A2.18) as:

$$e^{-V(X)} = \sum_{r=1}^s \sum_{X_r \subset X} \sum_{X_2 \dots X_{r-1}} \sum_{T \text{ on } X_r} (-1)^{r-1} \prod_{\ell \in T} V_\ell \int_0^1 dt_1 \dots \int_0^1 dt_{r-1} \left(\prod_{\ell \in T} \frac{\prod_{k=1}^{r-1} t_k(\ell)}{t_n(\ell)} \right) e^{-W_{X_r}(X_1, \dots, X_{r-1}; t_1, \dots, t_{r-1})} e^{-V(X \setminus X_r)} \quad (\text{A2.22})$$

where “ T on X_r ” means that T is an anchored tree for the clusters P_j with $j \in X_r$.

Let us define

$$K(X_r) = \sum_{X_2 \dots X_{r-1}} \sum_{T \text{ on } X_r} \prod_{\ell \in T} V_\ell \int_0^1 dt_1 \dots \int_0^1 dt_{r-1} \left(\prod_{\ell \in T} \frac{\prod_{k=1}^{r-1} t_k(\ell)}{t_n(\ell)} \right) e^{-W_{X_r}(X_1, \dots, X_{r-1}; t_1, \dots, t_{r-1})}, \quad (\text{A2.23})$$

so that (A2.22) can be rewritten as

$$e^{-V(X)} = \sum_{\substack{Y \subset X \\ Y \ni \{1\}}} (-1)^{|Y|-1} K(Y) e^{-V(X \setminus Y)}, \quad (\text{A2.24})$$

and, iterating,

$$e^{-V(X)} = \sum_{Q_1, \dots, Q_m} (-1)^{|X|} (-1)^m \prod_{q=1}^m K(Q_q). \quad (\text{A2.25})$$

The sets Q_1, \dots, Q_m in (A2.25) are disjoint subsets of X , such that $\cup_{i=1}^m Q_i = X$.

Substituting (A2.25) in (A2.7), we find

$$\mathcal{E} \left(\prod_{j=1}^s \tilde{\phi}(P_j) \right) = \int \mathcal{D}\phi \sum_{(Q_1, \dots, Q_m)} (-1)^{s+m} \prod_{q=1}^m K(Q_q), \quad (\text{A2.26})$$

where the sum is over the partitions (Q_1, \dots, Q_m) of X . It is easy to realize that in the last equation $K(Q_q)$ (already defined in (A2.23)) can be rewritten as

$$K(Q) = \sum_{T \text{ on } Q} \sum_{\substack{X_2, \dots, X_{|Q|-1} \\ \text{fixed } T}} \prod_{\ell \in T} V_\ell \int_0^1 dt_1 \dots \int_0^1 dt_{|Q|-1} \cdot \prod_{\ell \in T} (t_{n'(\ell)} \dots t_{n(\ell)-1}) e^{-\sum_{\ell \in Q \times Q} t_{n'(\ell)} \dots t_{n(\ell)} V_\ell} \quad (\text{A2.27})$$

Moreover, we can also rewrite (A2.26) as:

$$\mathcal{E} \left(\prod_{j=1}^s \tilde{\phi}(P_j) \right) = \sum_{(Q_1, \dots, Q_m)} (-1)^{s+m} (-1)^\sigma \prod_{q=1}^m \int \mathcal{D}\phi_{Q_q} K(Q_q), \quad (\text{A2.28})$$

where $\mathcal{D}\phi_{Q_q} = \prod_{j \in Q_q} \prod_{f \in P_j} d\phi_{\mathbf{x}(f), \omega(f)}^{\alpha(f)}$ and $(-1)^\sigma$ is the sign of the permutation leading from the ordering of the fields in $\mathcal{D}\phi$ to the ones in $\prod_q \mathcal{D}\phi_{Q_q}$.

Let us now consider the well known relation:

$$\mathcal{E} \left(\prod_{j=1}^s \tilde{\phi}(P_j) \right) = \sum_{(Q_1, \dots, Q_m)} (-1)^\sigma \mathcal{E}^T \left(\tilde{\phi}(P_{j_{11}}), \dots, \tilde{\phi}(P_{j_{1|Q_1|}}) \right) \dots \mathcal{E}^T \left(\tilde{\phi}(P_{j_{m1}}), \dots, \tilde{\phi}(P_{j_{m|Q_m|}}) \right), \quad (\text{A2.29})$$

where the sum is over the partitions of $\{1, \dots, s\}$, $Q_q = \{j_{q1}, \dots, j_{q|Q_q|}\}$ and $(-1)^\sigma$ is the parity of the permutation leading to the ordering on the r.h.s. from the one on the l.h.s. (note that σ is the same as in (A2.28)). Comparing (A2.29) with (A2.28) we get:

$$\mathcal{E}^T(\tilde{\phi}(P_1), \dots, \tilde{\phi}(P_s)) = (-1)^{s+1} \sum_{T \text{ on } X} \int \mathcal{D}\phi \prod_{\ell \in T} V_\ell \int dP_T(\mathbf{t}) e^{-V(\mathbf{t})}, \quad (\text{A2.30})$$

where we defined:

$$dP_T(\mathbf{t}) = \sum_{\substack{\mathbf{x}_2, \dots, \mathbf{x}_{s-1} \\ \text{fixed } T}} \prod_{\ell \in T} (t_{n'(\ell)} \dots t_{n(\ell)-1}) \prod_{q=1}^{s-1} dt_q \quad (\text{A2.31})$$

and

$$V(\mathbf{t}) \equiv \sum_{\ell \in X \times X} t_{n'(\ell)} \dots t_{n(\ell)} V_\ell. \quad (\text{A2.32})$$

If in (A2.30) we integrate the Grassman fields appearing in the product

$$\prod_{\ell \in T} V_\ell = \prod_{(jj') \in T} (\bar{V}_{jj'} + \bar{V}_{jj'}) , \quad (\text{A2.33})$$

we obtain

$$\mathcal{E}^T(\tilde{\phi}(P_1), \dots, \tilde{\phi}(P_s)) = (-1)^{s+1} \sum_{T \text{ on } P} \alpha_T \prod_{\ell \in T} G_{f_\ell^1, f_\ell^2} \int \mathcal{D}^*(d\phi) \int dP_T(\mathbf{t}) e^{-V^*(\mathbf{t})}, \quad (\text{A2.34})$$

where $P = \cup_i P_i$, the sum $\sum_{T \text{ on } P}$ denotes the sum over the graphs whose elements are lines connecting pairs of distinct points $\mathbf{x}(f)$, $f \in P$ such that, if we identify all the points in the clusters P_j , $j = 1, \dots, s$, then T is a tree graph on X ; moreover α_T is a suitable sign and

$$\mathcal{D}^*(d\phi) = \prod_{\substack{f \in P \\ f \notin T}} d\phi_{\mathbf{x}(f), \omega(f)}^{\alpha(f)}, \quad V^*(\mathbf{t}) = \sum_{\ell \notin T} t_{n'(\ell)} \dots t_{n(\ell)} V_\ell. \quad (\text{A2.35})$$

The term

$$\int \mathcal{D}^*(d\phi) \int dP_T(\mathbf{t}) e^{-V^*(\mathbf{t})} \quad (\text{A2.36})$$

in (A2.34) is the Pfaffian of a suitable matrix $G^T(\mathbf{t})$, with elements

$$G_{f, f'}^T(\mathbf{t}) = t_{n'(\ell)} \dots t_{n(\ell)} G_{f, f'}, \quad (\text{A2.37})$$

where $\ell = (j(f)j(f'))$, $j(f) \in X$ is s.t. $f \in P_{j(f)}$ and $G_{f, f'}$ was defined in (A2.2). So (4.14) is proven, with $t_{j, j'} = t_{n'(jj')} \dots t_{n(jj')}$.

In order to complete the proof of the claims following (4.14) we must prove that $dP_T(\mathbf{t})$ is a normalized, positive and σ -additive measure, so it can be interpreted as a probability measure in $\mathbf{t} = (t_1, \dots, t_{s-1})$; and that, moreover, we can find a family of versors $\mathbf{u}_j \in \mathbb{R}^s$ such that $t_{j, j'} = \mathbf{u}_j \cdot \mathbf{u}_{j'}$.

So, let us conclude this Appendix by proving the following Lemma.

LEMMA A2.1 $dP_T(\mathbf{t})$ is a normalized, positive and σ -additive measure on the natural σ -algebra of $[0, 1]^{s-1}$. Moreover there exists a set of unit vectors $\mathbf{u}_j \in \mathbb{R}^s$, $j = 1, \dots, s$, such that $t_{j,j'} = \mathbf{u}_j \cdot \mathbf{u}_{j'}$.

PROOF - Let us denote by b_k the number of lines $\ell \in T$ exiting from the points $x(j, i)$, $j \in X_k$, such that $\ell \sim X_k$. Let us consider the integral

$$\sum_{\substack{X_2 \dots X_{s-1} \\ T \text{ fissato}}} \int_0^1 dt_1 \dots \int_0^1 dt_{s-1} \prod_{\ell \in T} (t_{n'(\ell)} \dots t_{n(\ell)-1}) = 1, \quad (\text{A2.38})$$

and note that, by construction, the parameter t_k inside the integral in the l.h.s. appears at the power $b_k - 1$. In fact any line intersecting ∂X_k contributes by a factor t_k , except for the line connecting X_k with the point in $X_{k+1} \setminus X_k$. See Fig. A2.2.

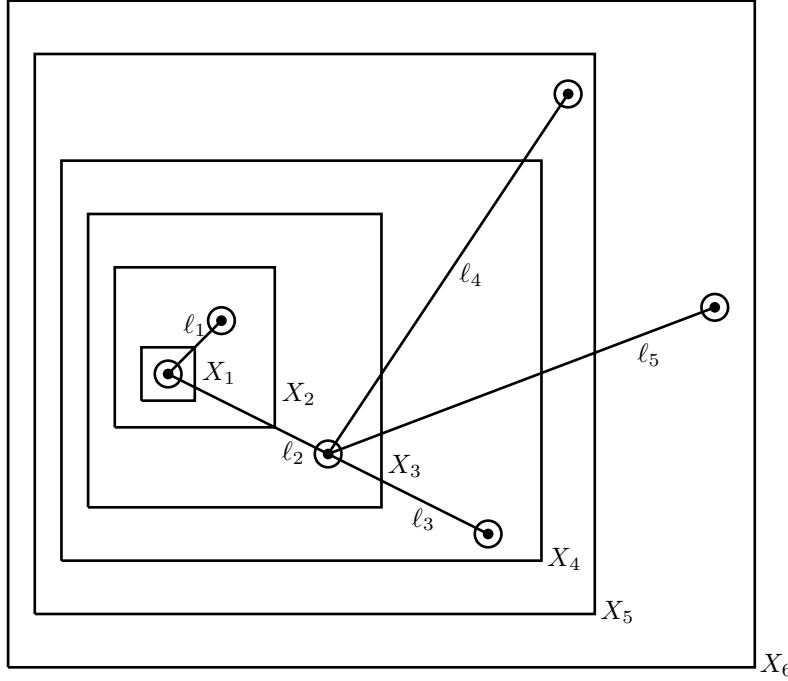


FIG. A2.1. The sets X_1, \dots, X_6 , the anchored tree T and the lines ℓ_1, \dots, ℓ_5 belonging to T . In the example, the coefficients b_1, \dots, b_5 are respectively equal to: 2, 1, 3, 2, 1.

Then

$$\prod_{\ell \in T} (t_{n'(\ell)} \dots t_{n(\ell)-1}) = \prod_{k=1}^{s-1} t_k^{b_k-1}, \quad (\text{A2.39})$$

and in (A2.38) the $s - 1$ integrations are independent. It holds:

$$\int_0^1 dt_1 \dots \int_0^1 dt_{s-1} \prod_{\ell \in T} (t_{n'(\ell)} \dots t_{n(\ell)-1}) = \prod_{k=1}^{s-1} \left(\int_0^1 dt_k t_k^{b_k-1} \right) = \prod_{k=1}^{s-1} \frac{1}{b_k}, \quad (\text{A2.40})$$

that is well defined, since $b_k \geq 1$, $k = 1, \dots, m - 2$. Moreover we can write:

$$\sum_{\substack{X_2 \dots X_{s-1} \\ T \text{ fixed}}} = \sum_{\substack{X_2 \\ T, X_1 \text{ fixed}}} \sum_{\substack{X_3 \\ T, X_1, X_2 \text{ fixed}}} \dots \sum_{\substack{X_{s-1} \\ T, X_1, \dots, X_{s-2} \text{ fixed}}}, \quad (\text{A2.41})$$

where the number of possible choices in summing over X_k , once that T and the sets X_1, \dots, X_{k-1} are fixed, is exactly b_{k-1} . In fact, if from X_{k-1} there are b_{k-1} exiting lines, then X_k is obtained by adding to X_{k-1} one of the b_{k-1} points connected to X_{k-1} through the tree lines. Then:

$$\sum_{\substack{X_2 \dots X_{s-1} \\ T \text{ fissato}}} 1 = b_1 \dots b_{s-2}, \quad (\text{A2.42})$$

and, recalling that $b_{s-1} = 1$,

$$\sum_{\substack{X_2 \dots X_{s-1} \\ T \text{ fissato}}} \int_0^1 dt_1 \dots \int_0^1 dt_{s-1} \prod_{\ell \in T} (t_{n'(\ell)} \dots t_{n(\ell)-1}) = \prod_{k=1}^{s-2} \frac{b_k}{b_k}, \quad (\text{A2.43})$$

yielding to $\int dP_T(\mathbf{t}) = 1$. The positivity and σ -additivity of $dP_T(\mathbf{t})$ is obvious by definition.

We are left with proving that we can find unit vectors $\mathbf{u}_j \in \mathbb{R}^s$ such that $t_{j,j'} = \mathbf{u}_j \cdot \mathbf{u}_{j'}$.

For this aim, let us introduce a family of unit vectors in \mathbb{R}^s defined as follows:

$$\begin{cases} \mathbf{u}_1 = \mathbf{v}_1, \\ \mathbf{u}_j = t_{j-1} \mathbf{u}_{j-1} + \mathbf{v}_j \sqrt{1 - t_{j-1}^2}, \quad j = 2, \dots, s, \end{cases} \quad (\text{A2.44})$$

where $\{\mathbf{v}_i\}_{i=1}^s$ is an orthonormal basis. Let us rename the sets P_i , $i = 1, \dots, s$ in such a way that $X_1 = \{1\}$, $X_2\{1, 2\}$, \dots , $X_{s-1} = \{1, \dots, s-1\}$. Then, for a given line (jj') , we have:

$$t_{j,j'} = t_{n'(jj')} \dots t_{n(jj')} = t_j \dots t_{j'-1} \quad (\text{A2.45})$$

From (A2.44) it follows

$$\mathbf{u}_j \cdot \mathbf{u}_{j'} = t_j \dots t_{j'-1} \quad (\text{A2.46})$$

as wanted. ■

Appendix A3. Gram–Hadamard inequality.

In this Appendix we prove Gram–Hadamard inequality, that is the bound (4.17).

Let $\mathbf{x}_1, \dots, \mathbf{x}_m$ be m vectors of a Hilbert space \mathcal{H} and let E be their span. We define the *Gram determinant* as

$$\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_m) \equiv \det \Gamma = \det \begin{pmatrix} (\mathbf{x}_1, \mathbf{x}_1) & \dots & (\mathbf{x}_1, \mathbf{x}_m) \\ \dots & \dots & \dots \\ (\mathbf{x}_m, \mathbf{x}_1) & \dots & (\mathbf{x}_m, \mathbf{x}_m) \end{pmatrix}, \quad (\text{A3.1})$$

where (\cdot, \cdot) denotes the inner product in \mathcal{H} . The following results hold.

LEMMA A3.1. *Given a Hilbert space \mathcal{H} and m vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ in \mathcal{H} , the Gram determinant (A3.1) satisfies*

$$\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_m) = 0, \quad (\text{A3.2})$$

if and only if the vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ are linearly independent. If the vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ are linearly independent then one has

$$\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_m) > 0. \quad (\text{A3.3})$$

PROOF - If the vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ are linearly dependent then there exist m coefficients c_1, \dots, c_m not all vanishing such that the vector $\sum_{j=1}^m c_j \mathbf{x}_j$ is vanishing. By considering its inner product with the vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$, we obtain the system

$$\begin{array}{ccccccc} c_1(\mathbf{x}_1, \mathbf{x}_1) & + & \dots & + & c_m(\mathbf{x}_1, \mathbf{x}_m) & = & 0 \\ \dots & & \dots & & \dots & & \dots \\ c_1(\mathbf{x}_m, \mathbf{x}_1) & + & \dots & + & c_m(\mathbf{x}_m, \mathbf{x}_m) & = & 0 \end{array} \quad (\text{A3.4})$$

which is an homogeneous system admitting a nontrivial solution: therefore the determinant of the matrix of the coefficients is zero, so implying (A3.2).

Vice versa if (A3.2) holds the system (A3.4) admits a nontrivial solution. If we multiply the m equations defining the system by c_1, \dots, c_m , respectively, then we sum them, we obtain

$$\|c_1 \mathbf{x}_1 + \dots + c_m \mathbf{x}_m\| = 0, \quad (\text{A3.5})$$

where $\|\cdot\|$ is the norm induced by the inner product (\cdot, \cdot) . Therefore the vector $\sum_{j=1}^m c_j \mathbf{x}_j$ has to be identically vanishing: as the coefficients c_1, \dots, c_m are not all vanishing, then the vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ have to be linearly dependent.

To prove (A3.3) consider a non trivial subset $S \subset E$, where E is the span of $\mathbf{x}_1, \dots, \mathbf{x}_m$, and set, for any $\mathbf{x} \in E$, $\mathbf{x} = \mathbf{x}_S + \mathbf{x}_N$, where $\mathbf{x}_S \in S$ and \mathbf{x}_N belonging to the orthogonal complement to S . We can write \mathbf{x}_N as $\mathbf{x}_N = c_1 \mathbf{x}_1 + \dots + c_p \mathbf{x}_p$, where $p < m$ and $p = n - \dim(S)$ (now we are assuming that the vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ are linearly independent). The vector

$$\det \begin{pmatrix} (\mathbf{x}_1, \mathbf{x}_1) & \dots & (\mathbf{x}_1, \mathbf{x}_p) & \mathbf{x}_1 \\ \dots & \dots & \dots & \dots \\ (\mathbf{x}_p, \mathbf{x}_1) & \dots & (\mathbf{x}_p, \mathbf{x}_p) & \mathbf{x}_p \\ (\mathbf{x}, \mathbf{x}_1) & \dots & (\mathbf{x}, \mathbf{x}_p) & \mathbf{x}_N \end{pmatrix} \quad (\text{A3.6})$$

is identically vanishing. In particular it follows that

$$\mathbf{x}_N = -\frac{1}{\det \Gamma} \det \begin{pmatrix} & \Gamma & \mathbf{x}_1 \\ & & \dots \\ (\mathbf{x}, \mathbf{x}_1) & \dots & (\mathbf{x}, \mathbf{x}_p) & 0 \end{pmatrix}, \quad (\text{A3.7})$$

and, analogously,

$$\mathbf{x}_S \equiv \mathbf{x} - \mathbf{x}_N = \frac{1}{\det \Gamma} \det \begin{pmatrix} & \Gamma & \mathbf{x}_1 \\ & & \dots \\ (\mathbf{x}, \mathbf{x}_1) & \dots & (\mathbf{x}, \mathbf{x}_p) & \mathbf{x} \end{pmatrix}, \quad (\text{A3.8})$$

so that

$$0 \leq h^2 \equiv (\mathbf{x}_S, \mathbf{x}) = \frac{1}{\det \Gamma} \det \begin{pmatrix} & \Gamma & (\mathbf{x}_1, \mathbf{x}) \\ & & \dots \\ (\mathbf{x}, \mathbf{x}_1) & \dots & (\mathbf{x}, \mathbf{x}_p) & (\mathbf{x}, \mathbf{x}) \end{pmatrix} = \frac{\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_p, \mathbf{x})}{\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_p)}. \quad (\text{A3.9})$$

By setting $\mathbf{x} \equiv \mathbf{x}_{p+1}$ and $h^2 = h_p^2$, we can write (A3.9) as

$$\frac{\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_p, \mathbf{x}_{p+1})}{\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_p)} = h_p^2 \geq 0, \quad (\text{A3.10})$$

where $\mathbf{x}_1, \dots, \mathbf{x}_p$ are p linearly independent vectors and \mathbf{x}_{p+1} is arbitrary. The sign $=$ in (A3.10) can hold if and only if \mathbf{x}_{p+1} is a linear combination of the vectors $\mathbf{x}_1, \dots, \mathbf{x}_p$ so that if $\mathbf{x}_1, \dots, \mathbf{x}_p, \mathbf{x}_{p+1}$ are linearly independent, then (A3.10) holds with the strict sign, *i.e.*

$$\frac{\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_p, \mathbf{x}_{p+1})}{\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_p)} = h_p^2 > 0. \quad (\text{A3.11})$$

As $\Gamma(\mathbf{x}_1) = (\mathbf{x}_1, \mathbf{x}_1) = \|\mathbf{x}_1\|^2 > 0$ for $\mathbf{x}_1 \neq 0$, (A3.11) implies (A3.3). ■

LEMMA A3.2 (HADAMARD INEQUALITY). *The Gram determinant satisfies the inequality*

$$\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_m) \leq \Gamma(\mathbf{x}_1) \dots \Gamma(\mathbf{x}_m), \quad (\text{A3.12})$$

where the sign $=$ holds if and only if the vectors are orthogonal to each other.

Proof. By (A3.11) and by using that $(\mathbf{x}_S, \mathbf{x}_S) \leq (\mathbf{x}, \mathbf{x}) = \Gamma(\mathbf{x})$, we have

$$\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{x}) \leq \Gamma(\mathbf{x}_1, \dots, \mathbf{x}_m) \Gamma(\mathbf{x}), \quad (\text{A3.13})$$

for any vectors $\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{x} \in E$. By iterating and recalling the arguments above (A3.12) follows. ■

Let $\mathbf{x}_1, \dots, \mathbf{x}_m$ be m linearly independent vectors in \mathcal{H} and E their span. Let $\{\underline{\varepsilon}_j\}_{j=1}^m$ an orthonormal basis

in E : set $x_{jk} = (\underline{\varepsilon}_j, \mathbf{x}_k)$, so that $\mathbf{x}_k = \sum_{j=1}^m x_{jk} \underline{\varepsilon}_j$, $k = 1, \dots, m$. Then

$$\begin{aligned}
 \Gamma(\mathbf{x}_1, \dots, \mathbf{x}_m) &= \det \begin{pmatrix} (\mathbf{x}_1, \mathbf{x}_1) & \dots & (\mathbf{x}_1, \mathbf{x}_m) \\ \dots & \dots & \dots \\ (\mathbf{x}_m, \mathbf{x}_1) & \dots & (\mathbf{x}_m, \mathbf{x}_m) \end{pmatrix} \\
 &= \det \begin{pmatrix} \sum_{ij} \bar{x}_{i1} x_{j1}(\underline{\varepsilon}_i, \underline{\varepsilon}_j) & \dots & \sum_{ij} \bar{x}_{i1} x_{jm}(\underline{\varepsilon}_i, \underline{\varepsilon}_j) \\ \dots & \dots & \dots \\ \sum_{ij} \bar{x}_{im} x_{j1}(\underline{\varepsilon}_i, \underline{\varepsilon}_j) & \dots & \sum_{ij} \bar{x}_{im} x_{jm}(\underline{\varepsilon}_i, \underline{\varepsilon}_j) \end{pmatrix} \\
 &= \det \begin{pmatrix} \sum_i \bar{x}_{i1} x_{i1} & \dots & \sum_i \bar{x}_{i1} x_{im} \\ \dots & \dots & \dots \\ \sum_i \bar{x}_{im} x_{i1} & \dots & \sum_i \bar{x}_{im} x_{im} \end{pmatrix} \\
 &= \det \begin{pmatrix} \bar{x}_{11} & \dots & \bar{x}_{m1} \\ \dots & \dots & \dots \\ \bar{x}_{1m} & \dots & \bar{x}_{mm} \end{pmatrix} \begin{pmatrix} x_{11} & \dots & x_{1m} \\ \dots & \dots & \dots \\ x_{m1} & \dots & x_{mm} \end{pmatrix} \\
 &= \det \bar{X}^T \det X = |\det X|^2,
 \end{aligned} \tag{A3.14}$$

where the matrix X is defined as

$$X = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1m} \\ x_{21} & x_{22} & \dots & x_{2m} \\ \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots & x_{mm} \end{pmatrix}. \tag{A3.15}$$

This yields that the Gram determinant (A3.12) can be written as

$$\Gamma(\mathbf{x}_1, \dots, \mathbf{x}_m) = |\det X|^2, \tag{A3.16}$$

so that from the lemma above the following result follows immediately.

LEMMA A3.3. *Given m linearly independent vectors of an Hilbert space \mathcal{H} and an orthonormal basis $\{\underline{\varepsilon}_j\}_{j=1}^m$ on their span, and defining the matrix X through (A3.14), one has*

$$|\det X|^2 \equiv |\det(\underline{\varepsilon}_i, \mathbf{x}_j)|^2 \leq \prod_{j=1}^m \|\mathbf{x}_j\|^2, \tag{A3.17}$$

where $(\underline{\varepsilon}_i, \mathbf{x}_j)$ stands for the matrix with entries $X_{ij} = (\underline{\varepsilon}_i, \mathbf{x}_j)$.

The lemma above is simply a reformulation of the preceeding Lemma: it implies the following inequality.

THEOREM A3.1 (GRAM-HADAMARD INEQUALITY). *Let $\{\mathbf{f}_j\}_{j=1}^m$ and $\{\mathbf{g}_j\}_{j=1}^m$ two families of m linearly independent vectors in an Euclidean space E , and let (\cdot, \cdot) an inner product in E and $\|\cdot\|$ the norm induced by that inner product. Then*

$$|\det(\mathbf{f}_i, \mathbf{g}_j)| \leq \prod_{j=1}^m \|\mathbf{f}_j\| \|\mathbf{g}_j\|, \tag{A3.18}$$

where $(\mathbf{f}_i, \mathbf{g}_j)$ stands for the $m \times m$ matrix with entries $(\mathbf{f}_i, \mathbf{g}_j)$.

PROOF - If $\{\mathbf{g}_j\}_{j=1}^m$ is an orthogonal basis in E (so that $\{\underline{\varepsilon}_j\}_{j=1}^m$, with $\underline{\varepsilon}_j = \|\mathbf{g}_j\|^{-1} \mathbf{g}_j$, is an orthonormal basis) then (A3.17) gives

$$|\det(\mathbf{g}_i, \mathbf{x}_j)| = |\det(\underline{\varepsilon}_i, \mathbf{x}_j)| \prod_{j=1}^m \|\mathbf{g}_j\| \leq \prod_{j=1}^m \|\mathbf{g}_j\| \|\mathbf{x}_j\|, \tag{A3.19}$$

Now consider the case in which the only conditions on the vectors $\{\mathbf{g}_j\}_{j=1}^m$ is that they are linearly independent. Set $\tilde{\mathbf{g}}_j = \|\mathbf{g}_j\|^{-1}\mathbf{g}_j$, so that $\|\tilde{\mathbf{g}}_j\|^2 = 1$, and define inductively the family of vectors

$$\begin{aligned}\tilde{\varepsilon}_1 &\equiv \tilde{\mathbf{g}}_1, \\ \tilde{\varepsilon}_2 &\equiv \frac{\tilde{\mathbf{g}}_2 - (\tilde{\mathbf{g}}_2, \tilde{\mathbf{g}}_1)\tilde{\mathbf{g}}_1}{1 - (\tilde{\mathbf{g}}_2, \tilde{\mathbf{g}}_1)^2},\end{aligned}\tag{A3.20}$$

and so on, in such a way that one has $(\tilde{\varepsilon}_i, \tilde{\varepsilon}_j) = \delta_{i,j}$. The basis $\{\varepsilon_1, \dots, \varepsilon_m\}$, with $\varepsilon_j = \tilde{\varepsilon}_j \ \forall j = 1, \dots, m$ is by construction an orthonormal basis.

If $c_2 = 1 - (\tilde{\mathbf{g}}_2, \tilde{\mathbf{g}}_1)^2$, with $0 \leq c_2 \leq 1$, one has

$$\tilde{\mathbf{g}}_2 = c_2 \tilde{\varepsilon}_2 + c_2 (\tilde{\mathbf{g}}_2, \tilde{\mathbf{g}}_1) \tilde{\mathbf{g}}_1, \tag{A3.21}$$

i.e. $\tilde{\mathbf{g}}_2 \sim c_2 \tilde{\varepsilon}_2$, if by \sim we mean that, by computing $\det(\tilde{\mathbf{g}}_i, \mathbf{f}_j)$, no difference is made by the fact that one has the vector $\tilde{\mathbf{g}}_2$ instead of $c_2 \tilde{\varepsilon}_2$: in fact the contributions arising from the remaining part in (A3.19) sum up to zero.

We can reason analogously for the terms with $j = 3, \dots, m$, and we find $\tilde{\mathbf{g}}_j \sim c_j \tilde{\varepsilon}_j$, where \sim is meant as above and the coefficients c_j are such that $0 \leq c_j \leq 1 \ \forall j = 1, \dots, m$. In conclusion:

$$\begin{aligned}|\det(\mathbf{g}_i, \mathbf{f}_j)| &= |\det(\tilde{\mathbf{g}}_i, \mathbf{f}_j)| \prod_{j=1}^m \|\mathbf{g}_j\| = |\det(\varepsilon_i, \mathbf{f}_j)| \prod_{j=1}^m c_j \|\mathbf{g}_j\| \\ &= \prod_{j=1}^m c_j \|\mathbf{g}_j\| \|\mathbf{f}_j\| \leq \prod_{j=1}^m \|\mathbf{g}_j\| \|\mathbf{f}_j\|, \end{aligned}\tag{A3.22}$$

so that (A3.18) follows. ■

Appendix A4. Proof of Lemma 5.3.

The propagators $g_{\underline{a}, \underline{a}'}^{(h)}(\mathbf{x})$ can be written in terms of the propagators $g_{\omega, \omega'}^{(j, h)}(\mathbf{x})$, $j = 1, 2$, see (5.20) and following lines; $g_{\omega, \omega'}^{(j, h)}(\mathbf{x})$ are given by

$$\begin{aligned} g_{\omega, \omega}^{(j, h)}(\mathbf{x} - \mathbf{y}) &= \\ &= \frac{2}{M^2} \sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x} - \mathbf{y})} \tilde{f}_h(\mathbf{k}) \frac{-i \sin k + \omega \sin k_0 + a_{h-1}^{-(j)}(\mathbf{k})}{\sin^2 k + \sin^2 k_0 + (\overline{m}_{h-1}^{(j)}(\mathbf{k}))^2 + \delta B_{h-1}^{(j)}(\mathbf{k})} \\ g_{\omega, -\omega}^{(j, h)}(\mathbf{x} - \mathbf{y}) &= \\ &= \frac{2}{M^2} \sum_{\mathbf{k}} e^{-i\mathbf{k}(\mathbf{x} - \mathbf{y})} \tilde{f}_h(\mathbf{k}) \frac{-i\omega \overline{m}_{h-1}^{(j)}(\mathbf{k})}{\sin^2 k + \sin^2 k_0 + (\overline{m}_{h-1}^{(j)}(\mathbf{k}))^2 + \delta B_{h-1}^{(j)}(\mathbf{k})}, \end{aligned} \quad (\text{A4.1})$$

where

$$\begin{aligned} a_{h-1}^{\omega(j)}(\mathbf{k}) &\stackrel{def}{=} -a_{h-1}^{\omega}(\mathbf{k}) + (-1)^j b_{h-1}^{\omega}(\mathbf{k}) \quad , \quad c_{h-1}^{(j)}(\mathbf{k}) \stackrel{def}{=} c_{h-1}(\mathbf{k}) + (-1)^j d_{h-1}(\mathbf{k}) \\ m_{h-1}^{(j)}(\mathbf{k}) &\stackrel{def}{=} \sigma_{h-1}(\mathbf{k}) + (-1)^j \mu_{h-1}(\mathbf{k}) \quad , \quad \overline{m}_{h-1}^{(j)}(\mathbf{k}) \stackrel{def}{=} m_{h-1}^{(j)}(\mathbf{k}) + c^{(j)}(\mathbf{k}) \\ \delta B_{h-1}^{(j)}(\mathbf{k}) &\stackrel{def}{=} \sum_{\omega} [a_{h-1}^{\omega(j)}(\mathbf{k})(i \sin k - \omega \sin k_0) + a_{h-1}^{\omega}(\mathbf{k}) a_{h-1}^{-\omega(j)}(\mathbf{k})/2] . \end{aligned} \quad (\text{A4.2})$$

In order to bound the propagators defined above, we need estimates on $\sigma_h(\mathbf{k})$, $\mu_h(\mathbf{k})$ and on the “corrections” $a_{h-1}^{\omega}(\mathbf{k})$, $b_{h-1}^{\omega}(\mathbf{k})$, $c_{h-1}(\mathbf{k})$, $d_{h-1}(\mathbf{k})$. As regarding $\sigma_h(\mathbf{k})$ and $\mu_h(\mathbf{k})$, it is easy to realize that, on the support of $f_h(\mathbf{k})$, for some c , $c^{-1}|\sigma_h| \leq |\sigma_{h-1}(\mathbf{k})| \leq c|\sigma_h|$ and $c^{-1}|\mu_h| \leq |\mu_{h-1}(\mathbf{k})| \leq c|\mu_h|$, see Proof of Lemma 2.6 in [BM]. Note also that, if $h \geq \bar{h}$, using the first two of (5.22), we have $\frac{|\sigma_h| + |\mu_h|}{\gamma_h} \leq 2C_1$. As regarding the corrections, using their iterative definition (5.14), the asymptotic estimates near $\mathbf{k} = \mathbf{0}$ of the corrections on scale $h = 1$ (see item (2) in Theorem 4.1) and the hypothesis (5.22), we easily find that, on the support of $f_h(\mathbf{k})$:

$$\begin{aligned} a_{h-1}^{\omega}(\mathbf{k}) &= O(\sigma_h \gamma^{(1-2c|\lambda|)h}) + O(\gamma^{(3-c|\lambda|^2)h}) \quad , \quad b_h^{\omega}(\mathbf{k}) = O(\mu_h \gamma^{(1-2c|\lambda|)h}) + O(\gamma^{(3-c|\lambda|^2)h}) , \\ c_h(\mathbf{k}) &= O(\gamma^{(2-c|\lambda|^2)h}) \quad , \quad d_h(\mathbf{k}) = O(\mu_h \gamma^{(2-2c|\lambda|)h}) . \end{aligned} \quad (\text{A4.3})$$

The bounds on the propagators follow from the remark that, as a consequence of the estimates discussed above, the denominators in (A4.1) are $O(\gamma^{2h})$ on the support of f_h .

Appendix A5. Proof of (5.42).

We have, by definition $\text{Pf } G = (2^k k!)^{-1} \sum_{\mathbf{p}} (-1)^{\mathbf{p}} G_{p(1)p(2)} \cdots G_{p(2k-1)p(2k)}$, where $\mathbf{p} = (p(1), \dots, p(2k))$ is a permutation of the indices $f \in J$ (we suppose $|J| = 2k$) and $(-1)^{\mathbf{p}}$ its sign.

If we apply $\mathcal{S}_1 = 1 - \mathcal{P}_0$ to $\text{Pf } G$ and we call $G_{f,f'}^0 \stackrel{\text{def}}{=} \mathcal{P}_0 G_{f,f'}$, we find that $\mathcal{S}_1 \text{Pf } G$ is equal to

$$\begin{aligned} \frac{1}{2^k k!} \sum_{\mathbf{p}} (-1)^{\mathbf{p}} \left[G_{p(1)p(2)} \cdots G_{p(2k-1)p(2k)} - G_{p(1)p(2)}^0 \cdots G_{p(2k-1)p(2k)}^0 \right] &= \frac{1}{2^k k!} \sum_{\mathbf{p}} (-1)^{\mathbf{p}} \sum_{j=1}^k \cdot \\ \cdot \left(G_{p(1)p(2)}^0 \cdots G_{p(2j-3)p(2j-2)}^0 \right) \mathcal{S}_1 G_{p(2j-1)p(2j)} &\left(G_{p(2j+1)p(2j+2)} \cdots G_{p(2k-1)p(2k)} \right), \end{aligned} \quad (\text{A5.1})$$

where in the last sum the meaningless factors must be put equal to 1. We rewrite the two sums over \mathbf{p} and j in the following way:

$$\sum_{\mathbf{p}} \sum_{j=1}^k = \sum_{j=1}^k \sum_{\substack{f_1, f_2 \in J \\ f_1 \neq f_2}} \sum_{J_1, J_2}^* \sum_{\mathbf{p}}^{**}, \quad (\text{A5.2})$$

where: the $*$ on the second sum means that the sets J_1 and J_2 are s.t. (f_1, f_2, J_1, J_2) is a partition of J ; the $**$ on the second sum means that $p(1), \dots, p(2j-2)$ belong to J_1 , $(p(2j-1), p(2j)) = (f_1, f_2)$ and $p(2j+1), \dots, p(2k)$ belong to J_2 . Using (A5.2) we can rewrite (A5.1) as

$$\begin{aligned} \mathcal{S}_1 \text{Pf } G &= \frac{1}{2^k k!} \sum_{j=1}^k \sum_{\substack{f_1, f_2 \in J \\ f_1 \neq f_2}} (-1)^{\pi} \mathcal{S}_1 G_{f_1, f_2} \sum_{J_1, J_2}^* \cdot \\ \cdot \sum_{\mathbf{p}_1, \mathbf{p}_2} (-1)^{\mathbf{p}_1 + \mathbf{p}_2} &\left(G_{p_1(1)p_1(2)}^0 \cdots G_{p_1(2k_1-1)p_1(2k_1)}^0 \right) \left(G_{p_2(1)p_2(2)} \cdots G_{p_2(2k_2-1)p_2(2k_2)} \right), \end{aligned} \quad (\text{A5.3})$$

where: $(-1)^{\pi}$ is the sign of the permutation leading from the ordering J to the ordering (f_1, f_2, J_1, J_2) ; \mathbf{p}_i , $i = 1, 2$ is a permutation of the labels in J_i (we suppose $|J_i| = 2k_i$) and $(-1)^{\mathbf{p}_i}$ is its sign. It is clear that (A5.3) is equivalent to (5.44).

Appendix A6. Vanishing of the Beta function.

In this Appendix we want to prove the first bound in (6.6), also called the *vanishing of the Luttinger model Beta function*; we reproduce the proof proposed in [BM1].

We will consider the reference model for our system, that is a model with propagator given by (6.4), a local quartic interaction and with both ultraviolet and infrared cutoffs. The model is similar (but not the same) to the Luttinger model, but it is not exactly solvable. However its Beta function, coinciding with the first term in the r.h.s. of (6.5), also coincide with the infrared part of the Luttinger model Beta function.

The reference model formally satisfies chiral gauge invariance, in the sense that, *neglecting the UV and IR cutoffs*, it is invariant under the transformations

$$\psi_{\mathbf{x},\omega}^{\pm} \rightarrow e^{\pm i\alpha_{\mathbf{x},\omega}} \psi_{\mathbf{x},\omega}^{\pm}, \quad (\partial_0 + i\omega\partial_1) \rightarrow (\partial_0 + i\omega\partial_1) + i[(\partial_0 + i\omega\partial_1)\alpha_{\mathbf{x},\omega}].$$

Using the invariance of the Schwinger functions generating functional under these transformations, one gets a hierarchy of *Ward identities*, which differ from the formal ones by terms which formally vanish when the cutoffs are removed. However these terms could give no trivial contributions to the correlation functions, because they must be included in the multiscale integration and the cutoffs must be removed after the integration procedure is finished. This is in fact the case, and the result can be expressed in terms of some correction identities, relating the corrections to the formal Ward identities to the 2 or 4 legs Schwinger functions. The exact Ward identities differ from the formal ones, even when the UV and IR cutoffs are removed. This is called *breaking of chiral symmetry*.

The Ward identities together with the so called Dyson equation, allow to express λ_h in terms of λ and of Schwinger functions satisfying the “right” dimensional bounds. The conclusion will be that, keeping the UV cutoff fixed at scale 0, $\lambda_h = \lambda + O(\lambda^2)$, uniformly in $h < 0$. This implies that the IR cutoff can be removed and that the infrared part of the Beta function satisfies the first of (6.6). The bound (6.6) is an easy consequence of the bound $\lambda_h = \lambda + O(\lambda^2)$, and the proof of this will be presented below for completeness, see (A6.97)–(A6.101).

A6.1. The reference model

The reference model is defined by the interaction

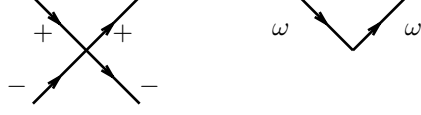
$$V(\psi) = \lambda \int d\mathbf{x} \psi_{\mathbf{x},+}^+ \psi_{\mathbf{x},+}^- \psi_{\mathbf{x},-}^+ \psi_{\mathbf{x},-}^- \quad (\text{A6.1})$$

where $\int d\mathbf{x}$ is a shorthand for “ $\sum_{\mathbf{x} \in \Lambda_M}$ ”, and by the free “measure”

$$P(d\psi) = \mathcal{N}^{-1} \mathcal{D}\psi \cdot \exp \left\{ -\frac{1}{M^2} \sum_{\omega=\pm 1} \sum_{\mathbf{k}} C_{h,0}(\mathbf{k}) (-ik_0 + \omega k) \hat{\psi}_{\mathbf{k},\omega}^+ \hat{\psi}_{\mathbf{k},\omega}^- \right\}, \quad (\text{A6.2})$$

where the summation over \mathbf{k} is over the momenta allowed by the antiperiodic boundary conditions, $\mathcal{N} = \prod_{\mathbf{k} \in \mathcal{D}} [(L\beta)^{-2} (-k_0^2 - k^2) C_{h,0}(\mathbf{k})^2]$ and $[C_{h,0}(\mathbf{k})]^{-1} \stackrel{\text{def}}{=} \sum_{k=h}^0 f_h(\mathbf{k}) \equiv \chi_{h,0}(\mathbf{k})$. We read the presence of $C_{h,0}(\mathbf{k})$ by saying that an ultraviolet cutoff on scale 0 and an infrared cutoff on scale h are imposed. We introduce the *generating functional*

$$\mathcal{W}(\phi, J) = \log \int P(d\psi) e^{-V(\psi) + \sum_{\omega} \int d\mathbf{x} [J_{\mathbf{x},\omega} \psi_{\mathbf{x},\omega}^+ \psi_{\mathbf{x},\omega}^- + \phi_{\mathbf{x},\omega}^+ \psi_{\mathbf{x},\omega}^- + \psi_{\mathbf{x},\omega}^+ \phi_{\mathbf{x},\omega}^-]}. \quad (\text{A6.3})$$

FIG. 1. Graphical representation of the interaction $V(\psi)$ and the density $\psi_{\mathbf{x},\omega}^+ \psi_{\mathbf{x},\omega}^-$

The interaction V and the density operators appearing at the exponent of (A6.3) can be represented as in Fig 1.

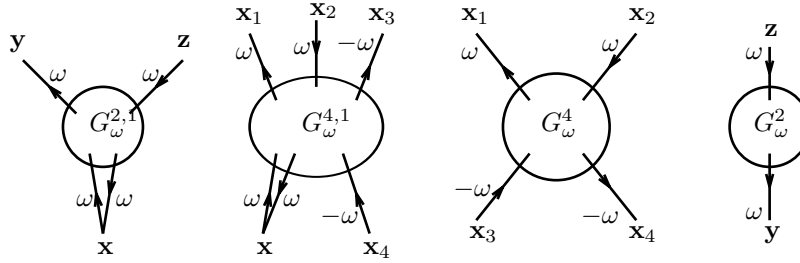
The Schwinger functions can be obtained by functional derivatives of (A6.3); for instance

$$G_{\omega}^{2,1}(\mathbf{x}; \mathbf{y}, \mathbf{z}) = \frac{\partial}{\partial J_{\mathbf{x},\omega}} \frac{\partial^2}{\partial \phi_{\mathbf{y},+}^+ \partial \phi_{\mathbf{z},+}^-} \mathcal{W}(\phi, J)|_{\phi=J=0}, \quad (\text{A6.4})$$

$$G_{\omega}^{4,1}(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \frac{\partial}{\partial J_{\mathbf{x},\omega}} \frac{\partial^2}{\partial \phi_{\mathbf{x}_1,\omega}^+ \partial \phi_{\mathbf{x}_2,\omega}^-} \frac{\partial^2}{\partial \phi_{\mathbf{x}_3,-\omega}^+ \partial \phi_{\mathbf{x}_4,-\omega}^-} \mathcal{W}(\phi, J)|_{\phi=J=0}, \quad (\text{A6.5})$$

$$G_{\omega}^4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \frac{\partial^2}{\partial \phi_{\mathbf{x}_1,\omega}^+ \partial \phi_{\mathbf{x}_2,\omega}^-} \frac{\partial^2}{\partial \phi_{\mathbf{x}_3,-\omega}^+ \partial \phi_{\mathbf{x}_4,-\omega}^-} \mathcal{W}(\phi, J)|_{\phi=J=0}, \quad (\text{A6.6})$$

$$G_{\omega}^2(\mathbf{y}, \mathbf{z}) = \frac{\partial^2}{\partial \phi_{\mathbf{y},\omega}^+ \partial \phi_{\mathbf{z},\omega}^-} \mathcal{W}(\phi, J)|_{\phi=J=0}. \quad (\text{A6.7})$$

FIG. 2. Graphical representation of the Schwinger functions $G_{\omega}^{2,1}, G_{\omega}^{4,1}, G_{\omega}^4, G_{\omega}^2$.

The generating functional and the Schwinger functions introduced above can be studied by a multiscale analysis similar to that described in Chapter 5, with some complications, due to the presence of the density fields J , and, from the other side, with some simplifications, due to the absence of mass terms in our action (it can be easily seen by symmetry that mass terms analogue of F_{σ} or F_{μ} cannot even be generated by the multiscale expansion). We will sketch the expansion below.

A6.2. The Dyson equation.

Let us consider the four legs Schwinger function G_{ω}^4 in (A6.6), computed at a momentum scale $= h$, which

is proportional to λ_h , as it is easy to realize. Since we want to connect λ_h with the “bare coupling” λ , it is natural to write a *Dyson* equation for \hat{G}^4 :

$$-\hat{G}_+^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \lambda \hat{g}_-(\mathbf{k}_4) \left[\hat{G}_-^2(\mathbf{k}_3) \hat{G}_+^{2,1}(\mathbf{k}_1 - \mathbf{k}_2, \mathbf{k}_1, \mathbf{k}_2) + \frac{1}{M^2} \sum_{\mathbf{p}} G_+^{4,1}(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) \right], \quad (\text{A6.8})$$

relating the correlations in (A6.4),(A6.5),(A6.6),(A6.7); see Fig. 3.

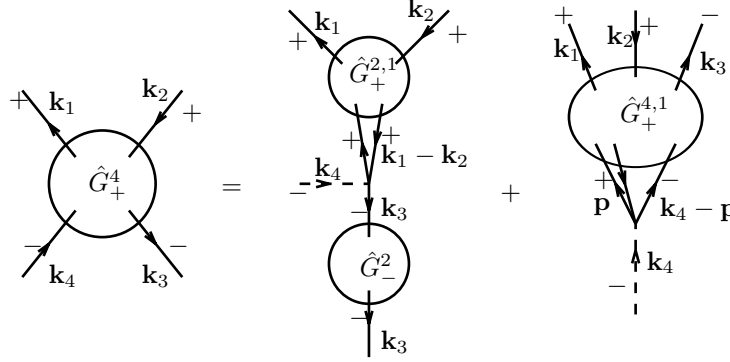


FIG. 3. Graphical representation of the Dyson equation (A6.8); the dotted line represents the “bare” propagator $g(\mathbf{k}_4)$

The Dyson equation can be derived as follows.

We define

$$G_\omega^{4,1}(\mathbf{z}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \langle \rho_{\mathbf{z}, \omega}; \psi_{\mathbf{x}_1, +}^-; \psi_{\mathbf{x}_2, +}^+; \psi_{\mathbf{x}_3, -}^-; \psi_{\mathbf{x}_4, -}^+ \rangle^T, \quad (\text{A6.9})$$

$$G_+^4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \langle \psi_{\mathbf{x}_1, +}^-; \psi_{\mathbf{x}_2, +}^+; \psi_{\mathbf{x}_3, -}^-; \psi_{\mathbf{x}_4, -}^+ \rangle^T, \quad (\text{A6.10})$$

where

$$\rho_{\mathbf{x}, \omega} = \psi_{\mathbf{x}, \omega}^+ \psi_{\mathbf{x}, \omega}^-. \quad (\text{A6.11})$$

Moreover, we shall denote by $\hat{G}_{+\omega}^{4,1}(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ and $\hat{G}_+^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ the corresponding Fourier transforms, deprived of the momentum conservation delta. Note that, if the ψ^+ momenta are interpreted as “incoming momenta” in the usual graph pictures, then the ψ^- momenta are “outgoing momenta”; our definition of Fourier transform is such that even \mathbf{p} , the momentum associated with the ρ field, is an ingoing momentum. Hence, the momentum conservation implies that $\mathbf{k}_1 + \mathbf{k}_3 = \mathbf{k}_2 + \mathbf{k}_4 + \mathbf{p}$, in the case of $\hat{G}_\omega^{4,1}(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ and $\mathbf{k}_1 + \mathbf{k}_3 = \mathbf{k}_2 + \mathbf{k}_4$ in the case of $\hat{G}_+^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$.

If $Z = \int P(d\psi) \exp\{-V(\psi)\}$ and $\langle \cdot \rangle$ denotes the expectation with respect to $Z^{-1} \int P(d\psi) \exp\{-V(\psi)\}$, by the definition of truncated expectation it follows:

$$G_+^4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \langle \psi_{\mathbf{x}_1, +}^-; \psi_{\mathbf{x}_2, +}^+; \psi_{\mathbf{x}_3, -}^-; \psi_{\mathbf{x}_4, -}^+ \rangle = -G_+^2(\mathbf{x}_1, \mathbf{x}_2) G_-^2(\mathbf{x}_3, \mathbf{x}_4), \quad (\text{A6.12})$$

where we used the fact that $\langle \psi_{\mathbf{x}, \omega}^- \psi_{\mathbf{y}, -\omega}^+ \rangle = 0$.

Let $g_\omega(\mathbf{x})$ be the free propagator, whose Fourier transform is $g_\omega(\mathbf{k}) = \chi_{h,0}(\mathbf{k})/(-ik_0 + \omega k)$. Then, we can write the last equation as

$$\begin{aligned} G_+^4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) &= -\lambda \int d\mathbf{z} g_-(\mathbf{z} - \mathbf{x}_4) \langle \psi_{\mathbf{x}_1, +}^-; \psi_{\mathbf{x}_2, +}^+; \psi_{\mathbf{x}_3, -}^-; \psi_{\mathbf{z}, -}^+; \psi_{\mathbf{z}, +}^- \rangle + \\ &+ \lambda G_+^2(\mathbf{x}_1, \mathbf{x}_2) \int d\mathbf{z} g_-(\mathbf{z} - \mathbf{x}_4) \langle \psi_{\mathbf{x}_3, -}^-; \psi_{\mathbf{z}, -}^+; \psi_{\mathbf{z}, +}^- \rangle = \\ &= -\lambda \int d\mathbf{z} g_{-1}(\mathbf{z} - \mathbf{x}_4) \langle [\psi_{\mathbf{x}_1, +}^-; \psi_{\mathbf{x}_2, +}^+]; [\psi_{\mathbf{x}_3, -}^-; \psi_{\mathbf{z}, -}^+; \psi_{\mathbf{z}, +}^-] \rangle^T. \end{aligned} \quad (\text{A6.13})$$

Again, by definition of truncated expectations, we have:

$$\langle \psi_{\mathbf{x}_1,+}^-; \psi_{\mathbf{x}_2,+}^+; \rho_{\mathbf{z},+} \rangle^T = \langle \psi_{\mathbf{x}_1,+}^- \psi_{\mathbf{x}_2,+}^+ \rho_{\mathbf{z},+} \rangle - \langle \psi_{\mathbf{x}_1,+}^- \psi_{\mathbf{x}_2,+}^+ \rangle \langle \rho_{\mathbf{z},+} \rangle, \quad (\text{A6.14})$$

and

$$\begin{aligned} & \langle \rho_{\mathbf{z},+}; \psi_{\mathbf{x}_1,+}^-; \psi_{\mathbf{x}_2,+}^+; \psi_{\mathbf{x}_3,-}^-; \psi_{\mathbf{z},-}^+ \rangle^T = \langle \rho_{\mathbf{z},+} \psi_{\mathbf{x}_1,+}^- \psi_{\mathbf{x}_2,+}^+ \psi_{\mathbf{x}_3,-}^- \psi_{\mathbf{z},-}^+ \rangle - \\ & - \langle \rho_{\mathbf{z},+} \psi_{\mathbf{x}_1,+}^- \psi_{\mathbf{x}_2,+}^+ \rangle \langle \psi_{\mathbf{x}_3,-}^- \psi_{\mathbf{z},-}^+ \rangle - \langle \rho_{\mathbf{z},+} \rangle \langle \psi_{\mathbf{x}_1,+}^- \psi_{\mathbf{x}_2,+}^+ \psi_{\mathbf{x}_3,-}^- \psi_{\mathbf{z},-}^+ \rangle - \\ & - \langle \psi_{\mathbf{x}_1,+}^- \psi_{\mathbf{x}_2,+}^+ \rangle \langle \rho_{\mathbf{z},+} \psi_{\mathbf{x}_3,-}^- \psi_{\mathbf{z},-}^+ \rangle + 2 \langle \rho_{\mathbf{z},+} \rangle \langle \psi_{\mathbf{x}_1,+}^- \psi_{\mathbf{x}_2,+}^+ \rangle \langle \psi_{\mathbf{x}_3,-}^- \psi_{\mathbf{z},-}^+ \rangle. \end{aligned} \quad (\text{A6.15})$$

Using the last two equations, together with (A6.12), we can rewrite (A6.13) as:

$$\begin{aligned} -G_+^4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) &= \lambda \int d\mathbf{z} g_-(\mathbf{z} - \mathbf{x}_4) \langle \psi_{\mathbf{x}_1,+}^-; \psi_{\mathbf{x}_2,+}^+; \rho_{\mathbf{z},+} \rangle^T \langle \psi_{\mathbf{x}_3,-}^- \psi_{\mathbf{z},-}^+ \rangle + \\ &+ \lambda \int d\mathbf{z} g_-(\mathbf{z} - \mathbf{x}_4) \langle \rho_{\mathbf{z},+}; \psi_{\mathbf{x}_1,+}^-; \psi_{\mathbf{x}_2,+}^+; \psi_{\mathbf{x}_3,-}^-; \psi_{\mathbf{z},-}^+ \rangle^T + \\ &+ \lambda \int d\mathbf{z} g_-(\mathbf{z} - \mathbf{x}_4) \langle \psi_{\mathbf{x}_1,+}^-; \psi_{\mathbf{x}_2,+}^+; \psi_{\mathbf{x}_3,-}^-; \psi_{\mathbf{z},-}^+ \rangle^T \langle \rho_{\mathbf{z},+} \rangle. \end{aligned} \quad (\text{A6.16})$$

The last addend is vanishing, since $\langle \rho_{\mathbf{z},\omega} \rangle = 0$ by the propagator parity properties. In terms of Fourier transforms, we get the *Dyson equation* (A6.8).

The l.h.s. of the Dyson equation computed at the cutoff scale is indeed proportional to the effective interaction λ_h (see (A6.34) below), while the r.h.s. is proportional to λ . *If one does not take into account cancellations in (A6.8)*, this equation only allows us to prove that $|\lambda_h| \leq C_h |\lambda|$, with C_h diverging as $h \rightarrow -\infty$. However, inspired by the analysis in the physical literature, see [DL][So][DM], we can try to express $\hat{G}_\omega^{2,1}$ and $\hat{G}_\omega^{4,1}$, in the r.h.s. of (A6.8), in terms of \hat{G}_ω^2 and \hat{G}_ω^4 by suitable *Ward identities* and *correction identities*.

A6.3. Ward identities and the first addend of (A6.8)

To begin with, we consider the first addend in the r.h.s. of the Dyson equation (A6.8). A remarkable identity relating $\hat{G}_+^{2,1}$ to \hat{G}_+^2 can be obtained by the chiral Gauge transformation $\psi_{\mathbf{x},+}^\pm \rightarrow e^{\pm i\alpha\mathbf{x}} \psi_{\mathbf{x},+}^\pm$, $\psi_{\mathbf{x},-}^\pm \rightarrow \psi_{\mathbf{x},-}^\pm$ in the generating functional (A6.3); one obtains the following identity, represented pictorially in Fig. 4, with $D_\omega(\mathbf{p}) = -ip_0 + \omega p$:

$$D_+(\mathbf{p}) \hat{G}_+^{2,1}(\mathbf{p}, \mathbf{k}) = G_+^2(\mathbf{q}) - G_+^2(\mathbf{k}) + \hat{\Delta}_+^{2,1}(\mathbf{p}, \mathbf{k}), \quad (\text{A6.17})$$

with $\hat{\Delta}_+^{2,1}$ the Fourier transform of $\Delta_+^{2,1}(\mathbf{x}; \mathbf{y}, \mathbf{z})$:

$$\Delta_+^{2,1}(\mathbf{x}; \mathbf{y}, \mathbf{z}) = \frac{1}{M^4} \sum_{\mathbf{k}, \mathbf{p}} e^{i\mathbf{p}\mathbf{x} - i\mathbf{k}\mathbf{y} + i(\mathbf{k}-\mathbf{p})\mathbf{z}} \hat{\Delta}_+^{2,1}(\mathbf{p}, \mathbf{k}) = \langle \psi_{\mathbf{y},+}^-; \psi_{\mathbf{z},+}^+; \delta T_{\mathbf{x},+} \rangle^T \quad (\text{A6.18})$$

and

$$\delta T_{\mathbf{x},\omega} = \frac{1}{M^2} \sum_{\mathbf{k}_+ \neq \mathbf{k}_-} e^{i(\mathbf{k}_+ - \mathbf{k}_-)\mathbf{x}} C_\omega(\mathbf{k}_+, \mathbf{k}_-) \hat{\psi}_{\mathbf{k}_+,\omega}^+ \hat{\psi}_{\mathbf{k}_-,\omega}^-, \quad (\text{A6.19})$$

$$C_\omega(\mathbf{k}^+, \mathbf{k}^-) = [C_{h,0}(\mathbf{k}^-) - 1] D_\omega(\mathbf{k}^-) - [C_{h,0}(\mathbf{k}^+) - 1] D_\omega(\mathbf{k}^+).$$

The above Ward identity can be derived as follows. Consider the chiral gauge transformation

$$\psi_{\mathbf{x},+}^\pm \rightarrow e^{i\pm\alpha\mathbf{x}} \psi_{\mathbf{x},+}^\pm, \quad \psi_{\mathbf{x},-}^\pm \rightarrow \psi_{\mathbf{x},-}^\pm, \quad (\text{A6.20})$$

and notice that $\mathcal{W}(\phi, J)$, as defined by (A6.3), is invariant under this change of variables. Then we can rewrite

$$\begin{aligned} \mathcal{W}(\phi, J) &= \log \int P(d\psi) \exp \left\{ - \int d\mathbf{x} \psi_{\mathbf{x},+}^+ \left(e^{i\alpha\mathbf{x}} D_+^{[h,0]} e^{-i\alpha\mathbf{x}} - D_+^{[h,0]} \right) \psi_{\mathbf{x},+}^- \right\} \cdot \\ &\cdot \exp \left\{ - V(\psi) + \int d\mathbf{x} \left[\sum_\omega J_{\mathbf{x},\omega} \psi_{\mathbf{x},\omega}^+ \psi_{\mathbf{x},\omega}^- + e^{-i\alpha\mathbf{x}} \phi_{\mathbf{x},+}^+ \psi_{\mathbf{x},+}^- + e^{i\alpha\mathbf{x}} \psi_{\mathbf{x},+}^+ \phi_{\mathbf{x},+}^- + \phi_{\mathbf{x},-}^+ \psi_{\mathbf{x},-}^- + \psi_{\mathbf{x},-}^+ \phi_{\mathbf{x},-}^- \right] \right\}. \end{aligned} \quad (\text{A6.21})$$

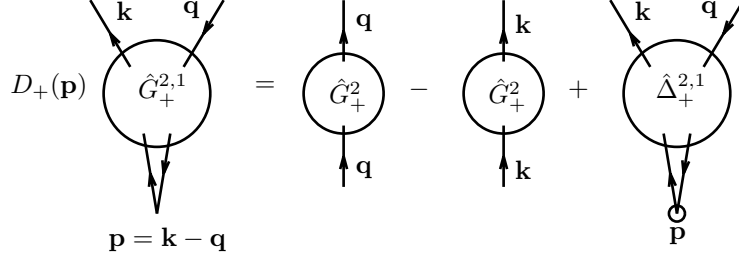


FIG. 4. Graphical representation of the Ward identity (A6.17); the small circle in $\hat{\Delta}_+^{2,1}$ represents the function C_+ of (A6.19).

where $D_\omega^{[h,0]}$, $\omega = \pm$, is the pseudo differential operator defined by

$$D_\omega^{[h,0]} \psi_{\mathbf{x},\omega}^\alpha = \frac{1}{M^2} \sum_{\mathbf{k}} e^{i\alpha \mathbf{k} \mathbf{x}} (i\alpha k_0 - \omega \alpha k) \hat{\psi}_{\mathbf{k},\omega}^\alpha, \quad D_\omega^{[h,0]} \psi_{\mathbf{x},-\omega}^\alpha = 0. \quad (\text{A6.22})$$

As we have just remarked, the l.h.s. of (A6.21) is independent of $\alpha_{\mathbf{x}}$. Then, by differentiating both sides w.r.t. $\alpha_{\mathbf{x}}$ and posing $\alpha_{\mathbf{x}} \equiv 0$, we find:

$$0 = \langle -D_+(\psi_{\mathbf{x},+}^+ \psi_{\mathbf{x},+}^-) - \delta T_{\mathbf{x},+} - \phi_{\mathbf{x},+}^+ \psi_{\mathbf{x},+}^- + \psi_{\mathbf{x},+}^+ \phi_{\mathbf{x},+}^- \rangle_{\phi,J}, \quad (\text{A6.23})$$

where

$$\langle \cdot \rangle_{\phi,J} \stackrel{\text{def}}{=} e^{-W(\phi,J)} \int P(d\psi) e^{-V(\psi) + \sum_\omega \int d\mathbf{x} [J_{\mathbf{x},\omega} \psi_{\mathbf{x},\omega}^+ \psi_{\mathbf{x},\omega}^- + \phi_{\mathbf{x},\omega}^+ \psi_{\mathbf{x},\omega}^- + \psi_{\mathbf{x},\omega}^+ \phi_{\mathbf{x},\omega}^-]}, \quad (\text{A6.24})$$

D_ω is defined as the Fourier transform of $D_\omega(\mathbf{p})$:

$$D_\omega(\psi_{\mathbf{x},\omega}^+ \psi_{\mathbf{x},\omega}^+) = \frac{1}{M^4} \sum_{\mathbf{p},\mathbf{k}} D_\omega(\mathbf{p}) e^{-i\mathbf{p}\mathbf{x}} \hat{\psi}_{\mathbf{k},\omega}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},\omega}^-, \quad (\text{A6.25})$$

and the corrections $\delta T_{\mathbf{x},+}$ and $C_\omega(\mathbf{k}_+, \mathbf{k}_-)$ were defined by (A6.19).

Now, differentiating (A6.23) w.r.t. $\phi_{\mathbf{y},+}^+$ and $\phi_{\mathbf{z},+}^-$ and setting $\phi = J = 0$, we find:

$$-D_+ G_+^{2,1}(\mathbf{x}; \mathbf{y}, \mathbf{z}) = \delta(\mathbf{x} - \mathbf{y}) G_+^2(\mathbf{x}, \mathbf{z}) - \delta(\mathbf{x} - \mathbf{z}) G_+^2(\mathbf{y}, \mathbf{x}) + \hat{\Delta}_+^{2,1}(\mathbf{x}; \mathbf{y}, \mathbf{z}), \quad (\text{A6.26})$$

whose Fourier transform gives (A6.17).

The use of Ward identities is to provide relations between Schwinger functions, but the correction terms (due to the cutoffs) substantially affect the Ward identities and apparently spoil them of their utility. However there are other remarkable relations connecting the correction terms to the Schwinger functions; such *correction identities* can be proved by performing a careful analysis of the renormalized expansion for the correction terms, and come out of the peculiar properties of the function $C_+(\mathbf{k}, \mathbf{k} - \mathbf{p})$, see next section. The *correction identity* for $\hat{\Delta}_+^{2,1}$ is the following, see Fig.5.

$$\hat{\Delta}_+^{2,1}(\mathbf{p}, \mathbf{k}) = D_+(\mathbf{p}) \left[\nu_+ \hat{G}_+^{2,1}(\mathbf{p}, \mathbf{k}) + \nu_- \frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} \hat{G}_-^{2,1}(\mathbf{p}, \mathbf{k}) + \hat{H}_+^{2,1}(\mathbf{p}, \mathbf{k}) \right] \quad (\text{A6.27})$$

where ν_+, ν_- are $O(\lambda)$ and weakly dependent on h , once we prove that λ_j is small enough for $j \geq h$, and $\hat{H}_+^{2,1}(\mathbf{p}, \mathbf{k}, \mathbf{q})$ can be obtained through the analogue of (A6.4), with $\mathcal{W}(\phi, J)$ replaced by

$$\mathcal{W}_\Delta(\phi, J) = \log \int P(d\psi) e^{-V(\psi) + \int d\mathbf{x} [J_{\mathbf{x},+} T_{\mathbf{x}} + \sum_{\omega} (-\nu_{\omega} J_{\mathbf{x},+} T_{\mathbf{x},\omega}^{\nu} + \phi_{\mathbf{x},\omega}^+ \psi_{\mathbf{x},\omega}^- + \psi_{\mathbf{x},\omega}^+ \phi_{\mathbf{x},\omega}^-)]}, \quad (\text{A6.28})$$

where

$$\begin{aligned} T_{\mathbf{x}} &= \frac{1}{M^4} \sum_{\mathbf{k}_+ \neq \mathbf{k}_-} e^{i(\mathbf{k}_+ - \mathbf{k}_-) \cdot \mathbf{x}} \frac{C_+(\mathbf{k}_+, \mathbf{k}_-)}{D_+(\mathbf{k}_+ - \mathbf{k}_-)} \hat{\psi}_{\mathbf{x},+}^+ \hat{\psi}_{\mathbf{x},+}^-, \\ T_{\mathbf{x},\omega}^{\nu} &= \frac{1}{M^4} \sum_{\mathbf{k}_+ \neq \mathbf{k}_-} e^{i(\mathbf{k}_+ - \mathbf{k}_-) \cdot \mathbf{x}} \frac{D_{\omega}(\mathbf{k}_+ - \mathbf{k}_-)}{D_+(\mathbf{k}_+ - \mathbf{k}_-)} \hat{\psi}_{\mathbf{x},\omega}^+ \hat{\psi}_{\mathbf{x},\omega}^-, \end{aligned} \quad (\text{A6.29})$$

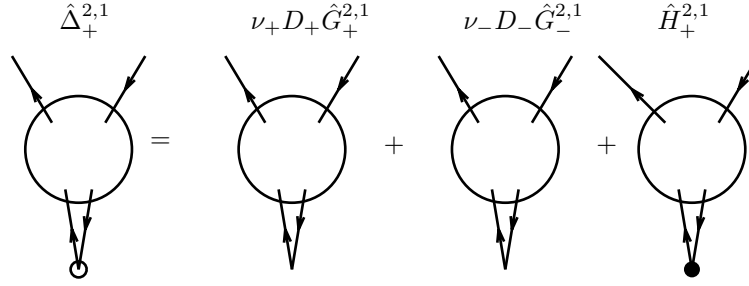


FIG. 5. Graphical representation of the correction identity (A6.27); the filled point in the last term represents $J_{\mathbf{x},+}(T_{\mathbf{x}} - \sum_{\omega} \nu_{\omega} T_{\mathbf{x},\omega}^{\nu})$.

The crucial point is that if ν_{\pm} are suitably chosen, $\hat{H}_+^{2,1}$, when computed for momenta at the cut-off scale, is $O(\gamma^{\vartheta h})$ smaller, with $0 < \vartheta < 1$ a positive constant, with respect to the first two addends of the r.h.s. of (A6.27). In other words the correction identity (A6.27) says that the correction term $\hat{\Delta}_+^{2,1}$, which is usually neglected in the physical literature, can be written in terms of the Schwinger functions $\hat{G}_+^{2,1}$ and $\hat{G}_-^{2,1}$ up to the exponentially smaller term $\hat{H}_+^{2,1}$.

Inserting the correction identity (A6.27) in the Ward identity (A6.17), we obtain the new identity

$$(1 - \nu_+) D_+(\mathbf{p}) \hat{G}_+^{2,1}(\mathbf{p}, \mathbf{k}, \mathbf{q}) - \nu_- D_-(\mathbf{p}) \hat{G}_-^{2,1}(\mathbf{p}, \mathbf{k}, \mathbf{q}) = \hat{G}_+^2(\mathbf{q}) - \hat{G}_+^2(\mathbf{k}) + \hat{H}_+^{2,1}(\mathbf{p}, \mathbf{k}, \mathbf{q}). \quad (\text{A6.30})$$

In the same way one can show that the formal Ward identity $D_-(\mathbf{p}) \hat{G}_-^{2,1}(\mathbf{p}, \mathbf{k}, \mathbf{q}) = \hat{G}_-^2(\mathbf{q}) - \hat{G}_-^2(\mathbf{k})$ becomes, if the cutoffs are taken into account:

$$(1 - \nu'_-) D_-(\mathbf{p}) \hat{G}_-^{2,1}(\mathbf{p}, \mathbf{k}, \mathbf{q}) - \nu'_+ D_+(\mathbf{p}) \hat{G}_+^{2,1}(\mathbf{p}, \mathbf{k}, \mathbf{q}) = \hat{H}_-^{2,1}(\mathbf{p}, \mathbf{k}, \mathbf{q}), \quad (\text{A6.31})$$

where, again $\nu'_{\pm} = O(\lambda)$ and $H_{\pm}^{2,1}$ satisfies a bound similar to that of $H_+^{2,1}$, when computed for momenta at the cutoff scale.

The identities (A6.30) and (A6.31) allow us to write $\hat{G}_+^{2,1}$ in terms of \hat{G}_{\pm}^2 and $\hat{H}_{\pm}^{2,1}$; one finds:

$$D_+(\mathbf{p}) \hat{G}_+^{2,1}(\mathbf{p}, \mathbf{k}) = \left[1 - \nu_+ - \frac{\nu_- \nu'_+}{1 - \nu'_-} \right]^{-1} \left\{ \hat{G}_+^2(\mathbf{q}) - \hat{G}_+^2(\mathbf{k}) + \frac{\nu_-}{1 - \nu'_-} \hat{H}_-^{2,1}(\mathbf{p}, \mathbf{k}) + \hat{H}_+^{2,1}(\mathbf{p}, \mathbf{k}) \right\}, \quad (\text{A6.32})$$

In order to bound $\hat{G}_+^{2,1}$, we can use the dimensional bounds for the two and four legs Schwinger functions, easily proved by repeating for $\mathcal{W}(\phi, J)$ an iterative construction similar to that exposed in Chapter 5 and

performing the bounds as explained in §5.5. The expansion involves the definition of a more involved localization operator, also acting on the kernels of the monomials involving the external fields and can be found in many review papers [BGPS][GM][BM]; it is very similar (and even simpler) to the expansion for $\mathcal{W}_\Delta(\phi, J)$, that will be described in next section.

The result we need is the following.

THEOREM A6.1 *There exists ε_0 such that, if $\bar{\lambda}_h \stackrel{\text{def}}{=} \max_{k>h} |\lambda_k| \leq \varepsilon_0$ and $|\bar{\mathbf{k}}| = \gamma^h$, then*

$$\hat{G}_\omega^{2,1}(2\bar{\mathbf{k}}, \bar{\mathbf{k}}) = -\frac{\gamma^{\eta_{2,1}h}}{Z_h D_\omega(\bar{\mathbf{k}})^2} [1 + O(\bar{\lambda}_h^2)] , \quad (\text{A6.33})$$

where $\eta_{2,1}(\lambda)$ is an exponent $O(\lambda)$ and

$$\hat{G}_\omega^2(\bar{\mathbf{k}}) = \frac{1}{Z_h D_\omega(\bar{\mathbf{k}})} [1 + O(\bar{\lambda}_h^2)] , \quad \hat{G}_+^4(\bar{\mathbf{k}}, -\bar{\mathbf{k}}, -\bar{\mathbf{k}}) = Z_h^{-2} |\bar{\mathbf{k}}|^{-4} [-\lambda_h + O(\bar{\lambda}_h^2)] . \quad (\text{A6.34})$$

Remarks

1 - The proof of Theorem A6.1 follows by a repetition of the estimates of Chapter 5. For some references: (A6.33) follows by the analysis in [BM2]; the first of (A6.34) can be proven as explained in [BGPS][GM]; the second (A6.34) follows as a combination of the first of (A6.34) and of the results in Chapter 5.

2 - A posteriori, it will result that $\eta_{2,1}(\lambda) = 0$. For the moment we just need (A6.33) to bound $|\hat{H}_+^{2,1}|$ with a constant times $\gamma^{\vartheta h}$ times the r.h.s. of (A6.33), as explained above, that is

$$|\hat{H}_+^{2,1}(2\bar{\mathbf{k}}, \bar{\mathbf{k}})| \leq C \gamma^{\vartheta h} \frac{\gamma^{\eta_{2,1}h}}{Z_h D_\omega(\bar{\mathbf{k}})^2} \leq C \gamma^{(\vartheta/2)h} \gamma^{-2h} \quad (\text{A6.35})$$

Substituting the preceding bounds into (A6.32), we soon find that $\hat{G}^{2,1}(\bar{\mathbf{p}}, \bar{\mathbf{k}})$, with $|\bar{\mathbf{p}}| = |\bar{\mathbf{k}}| = \gamma^h$, can be bounded as

$$|\hat{G}_\omega^{2,1}(\bar{\mathbf{p}}, \bar{\mathbf{k}})| \leq C \frac{\gamma^{-2h}}{Z_h} , \quad (\text{A6.36})$$

where, as in Theorem A6.1, we assumed $\bar{\lambda}_h \leq \varepsilon_0$.

Substituting the last bound into the first addend of (A6.8), with the arguments set on scale h , we soon find

$$\left| \lambda_{\hat{g}_-}(\mathbf{k}_4) \hat{G}_-^2(\mathbf{k}_3) \hat{G}_+^{2,1}(\mathbf{k}_1 - \mathbf{k}_2, \mathbf{k}_1, \mathbf{k}_2) \right| \leq C |\lambda| \gamma^{-h} \frac{\gamma^{-h}}{Z_h} \frac{\gamma^{-2h}}{Z_h} , \quad (\text{A6.37})$$

that is the “right” dimensional bound. In fact the l.h.s. of (A6.8) can be bounded as in (A6.34) so that, if we could neglect the second term in the r.h.s. (A6.8), we would soon find $|\lambda_h| \leq C|\lambda|$.

Aim of the next sections will be first to prove the correction identity (A6.27); then to describe a strategy which will allow us to find a “right” dimensional bound also for the second term in the r.h.s. (A6.8).

A6.4. The first correction identity

We start from the generating function (A6.28) and we perform iteratively the integration of the ψ variables, to be defined iteratively in the following way. After the fields $\psi^{(0)}, \dots, \psi^{(j)}$ have been integrated, we can write

$$e^{\mathcal{W}_\Delta(\phi, J)} = e^{-M^2 E_j} \int P_{\tilde{Z}_j, C_{h,j}}(d\psi^{[h,j]}) e^{-\mathcal{V}^{(j)}(\sqrt{Z_j} \psi^{[h,j]}) + K^{(j)}(\sqrt{Z_j} \psi^{[h,j]}, \phi, J)} , \quad (\text{A6.38})$$

with $\mathcal{V}^{(j)}(0) = 0$, $Z_j = \max_{\mathbf{k}} \tilde{Z}_j(\mathbf{k})$,

1) $P_{\tilde{Z}_j, C_{h,j}}(d\psi^{[h,j]})$ is the *effective Grassmannian measure at scale j* , equal to

$$P_{\tilde{Z}_j, C_{h,j}}(d\psi^{[h,j]}) = \prod_{\mathbf{k}: C_{h,j}(\mathbf{k}) > 0} \prod_{\omega=\pm 1} \frac{d\hat{\psi}_{\mathbf{k},\omega}^{[h,j]+} d\hat{\psi}_{\mathbf{k},\omega}^{[h,j]-}}{\mathcal{N}_j(\mathbf{k})} \cdot \exp \left\{ -\frac{1}{L\beta} \sum_{\mathbf{k}} C_{h,j}(\mathbf{k}) \tilde{Z}_j(\mathbf{k}) \sum_{\omega=\pm 1} \hat{\psi}_{\omega}^{[h,j]+} D_{\omega}(\mathbf{k}) \hat{\psi}_{\mathbf{k},\omega}^{[h,j]-} \right\}, \quad (\text{A6.39})$$

$$\mathcal{N}_j(\mathbf{k}) = (L\beta)^{-1} C_{h,j}(\mathbf{k}) \tilde{Z}_j(\mathbf{k}) [-k_0^2 - k^2]^{1/2}, \quad (\text{A6.40})$$

$$C_{h,j}(\mathbf{k})^{-1} = \sum_{r=h}^j f_r(\mathbf{k}) \equiv \chi_{h,j}(\mathbf{k}) \quad , \quad D_{\omega}(\mathbf{k}) = -ik_0 + \omega k; \quad (\text{A6.41})$$

2) the *effective potential on scale j* , $\mathcal{V}^{(j)}(\psi)$, is a sum of monomial of Grassmannian variables multiplied by suitable kernels, as in (5.5). The localization operator acts on the kernels of $\mathcal{V}^{(j)}(\psi)$ as described in §5.2. Note however that in the present case (*i.e.* for the reference model) the terms proportional to F_{σ} and $F_{\bar{\sigma}}$ are automatically vanishing, by symmetry: only the terms proportional to F_{λ} and F_{ζ} survive.

3) the *effective source term at scale j* , $K^{(j)}(\sqrt{Z_j}\psi, \phi, J)$, is a sum of monomials of Grassmannian variables and ϕ^{\pm}, J field, with at least one ϕ^{\pm} or one J field; we shall write it in the form

$$K^{(j)}(\sqrt{Z_j}\psi, \phi, J) = \mathcal{B}_{\phi}^{(j)}(\sqrt{Z_j}\psi) + K_J^{(j)}(\sqrt{Z_j}\psi) + W_R^{(j)}(\sqrt{Z_j}\psi, \phi, J), \quad (\text{A6.42})$$

where $\mathcal{B}_{\phi}^{(j)}(\psi)$ and $K_J^{(j)}(\psi)$ denote the sums over the terms containing only one ϕ or J field, respectively.

Of course (A6.38) is true for $j = 0$, with

$$\begin{aligned} \tilde{Z}_0(\mathbf{k}) &= 1, \quad E_0 = 0, \quad \mathcal{V}^{(0)}(\psi) = V(\psi), \quad W_R^{(0)} = 0, \\ \mathcal{B}_{\phi}^{(0)}(\psi) &= \sum_{\omega} \int d\mathbf{x} [\phi_{\mathbf{x},\omega}^{+} \psi_{\mathbf{x},\omega}^{-} + \psi_{\mathbf{x},\omega}^{+} \phi_{\mathbf{x},\omega}^{-}], \quad K_J^{(0)}(\psi) = \int d\mathbf{x} J_{\mathbf{x},+} \left(T_{\mathbf{x}} - \sum_{\omega} \nu_{\omega} T_{\mathbf{x},\omega}^{\nu} \right). \end{aligned} \quad (\text{A6.43})$$

Let us now assume that (A6.38) is satisfied for a certain $j \leq 0$ and let us show that it holds also with $j-1$ in place of j .

In order to perform the integration corresponding to $\psi^{(j)}$, we write the effective potential and the effective source as sum of two terms, according to the following rules.

We split the effective potential $\mathcal{V}^{(j)}$ as $\mathcal{L}\mathcal{V}^{(j)} + \mathcal{R}\mathcal{V}^{(j)}$, with \mathcal{L} acting on $\mathcal{V}^{(j)}$ as explained in §5.2.

Analogously we write $K^{(j)} = \mathcal{L}K^{(j)} + \mathcal{R}K^{(j)}$, $\mathcal{R} = 1 - \mathcal{L}$, according to the following definition. First of all, we put $\mathcal{L}W_R^{(j)} = W_R^{(j)}$.

Let us consider now $\mathcal{B}_{\phi}^{(j)}(\sqrt{Z_j}\psi)$; we want to show that, by a suitable choice of the localization procedure, if $j \leq -1$, it can be written in the form

$$\begin{aligned} \mathcal{B}_{\phi}^{(j)}(\sqrt{Z_j}\psi) &= \sum_{\omega} \sum_{i=j+1}^0 \int d\mathbf{x} d\mathbf{y} \cdot \\ &\cdot \left[\phi_{\mathbf{x},\omega}^{+} g_{\omega}^{Q,(i)}(\mathbf{x} - \mathbf{y}) \frac{\partial}{\partial \psi_{\mathbf{y},\omega}^{+}} \mathcal{V}^{(j)}(\sqrt{Z_j}\psi) + \frac{\partial}{\partial \psi_{\mathbf{y},\omega}^{-}} \mathcal{V}^{(j)}(\sqrt{Z_j}\psi) g_{\omega}^{Q,(i)}(\mathbf{y} - \mathbf{x}) \phi_{\mathbf{x},\omega}^{-} \right] + \\ &+ \sum_{\omega} \int \frac{d\mathbf{k}}{(2\pi)^2} \left[\hat{\psi}_{\mathbf{k},\omega}^{[h,j]+} \hat{Q}_{\omega}^{(j+1)}(\mathbf{k}) \hat{\phi}_{\mathbf{k},\omega}^{-} + \hat{\phi}_{\mathbf{k},\omega}^{+} \hat{Q}_{\omega}^{(j+1)}(\mathbf{k}) \hat{\psi}_{\mathbf{k},\omega}^{[h,j]-} \right], \end{aligned} \quad (\text{A6.44})$$

where $\hat{g}_{\omega}^{Q,(i)}(\mathbf{k}) = \hat{g}_{\omega}^{(i)}(\mathbf{k}) \hat{Q}_{\omega}^{(i)}(\mathbf{k})$, with

$$\hat{g}_{\omega}^{(j)}(\mathbf{k}) = \frac{1}{Z_{j-1}} \frac{\tilde{f}_j(\mathbf{k})}{D_{\omega}(\mathbf{k})}, \quad (\text{A6.45})$$

$\tilde{f}_j(\mathbf{k}) = f_j(\mathbf{k})Z_{j-1}[\tilde{Z}_{j-1}(\mathbf{k})]^{-1}$ and $Q_\omega^{(j)}(\mathbf{k})$ defined inductively by the relations

$$\hat{Q}_\omega^{(j)}(\mathbf{k}) = \hat{Q}_\omega^{(j+1)}(\mathbf{k}) - z_j Z_j D_\omega(\mathbf{k}) \sum_{i=j+1}^0 \hat{g}_\omega^{Q,(i)}(\mathbf{k}), \quad \hat{Q}_\omega^{(0)}(\mathbf{k}) = 1. \quad (\text{A6.46})$$

Note that $\hat{g}_\omega^{(j)}(\mathbf{k})$ does not depend on the infrared cutoff for $j > h$ and that (even for $j = h$) $\hat{g}^{(j)}(\mathbf{k})$ is of size $Z_{j-1}^{-1}\gamma^{-j}$, see discussion in §3 of [BM3], after eq. (60). Moreover the propagator $\hat{g}_\omega^{Q,(i)}(\mathbf{k})$ is equivalent to $\hat{g}_\omega^{(i)}(\mathbf{k})$, as concerns the dimensional bounds.

The \mathcal{L} operation for $\mathcal{B}_\phi^{(j)}$ is defined by decomposing $\mathcal{V}^{(j)}$ in the r.h.s. of (A6.44) as $\mathcal{LV}^{(j)} + \mathcal{RV}^{(j)}$.

Finally we have to define \mathcal{L} for $K_J^{(j)}(\sqrt{Z_j}\psi)$. It is easy to see that the field J is equivalent, from the point of view of dimensional considerations, to two ψ fields. Hence, the only terms which need to be renormalized are those of second order in ψ , which are indeed marginal; let us denote their sum with $K_J^{(j,2)}$. Let us start with defining the \mathcal{L} operation on $K_J^{(0)}$ as the identity. Let us now analyze the structure of $K_J^{(-1,2)}(\sqrt{Z_{-1}}\psi^{[h,-1]})$, as it appears after integrating the $\psi^{(0)}$ field and rescaling $\psi^{[h,-1]}$. We have

$$K_J^{(-1,2)}(\psi) = \frac{1}{Z_{-1}} \int d\mathbf{x} J_{\mathbf{x},+} \left\{ T_{\mathbf{x}} + \sum_{\omega} \int d\mathbf{y} d\mathbf{z} [F_{2,+, \omega}^{(-1)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) + F_{1, +}^{(-1)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \delta_{+, \omega}] \psi_{\mathbf{y}, \omega}^+ \psi_{\mathbf{z}, \omega}^- \right\} \quad (\text{A6.47})$$

$F_{2,+, \omega}^{(-1)}$ denotes the sum of all Feynman diagrams containing a $T_{\mathbf{x}, \omega}^\nu$ vertex or those obtained by contracting both ψ fields of a $T_{\mathbf{x}}$ vertex (the index ω refers to the ω index of the two left-over external ψ fields). $F_{1, +}^{(-1)}$ represents the sum over the diagrams built by leaving external one of these fields of $T_{\mathbf{x}}$.

Now, if $\mathbf{S}_\omega^{(0)}$ is defined as in Appendix A7, it is easy to see that the Fourier transform of $F_{2,+, \omega}^{(-1)}$ can be written as

$$\hat{F}_{2,+, \omega}^{(-1)}(\mathbf{k}_+, \mathbf{k}_-) = \frac{\mathbf{p}}{D_+(\mathbf{p})} \int d\tilde{\mathbf{k}}_+ \mathbf{S}_+^{(0)}(\tilde{\mathbf{k}}_+, \tilde{\mathbf{k}}_+ - \mathbf{p}) G_{+, \omega}^{(-1)}(\tilde{\mathbf{k}}_+, \mathbf{k}_+, \mathbf{k}_-), \quad (\text{9.48})$$

where $\mathbf{p} = \mathbf{k}_+ - \mathbf{k}_-$ and $G_{+, \omega}^{(-1)}(\tilde{\mathbf{k}}_+, \mathbf{k}_+, \mathbf{k}_-)$ is of the form

$$G_{+, \omega}^{(-1)}(\tilde{\mathbf{k}}_+, \mathbf{k}_+, \mathbf{k}_-) = G_0(\tilde{\mathbf{k}}_+, \mathbf{k}_+, \mathbf{k}_-) + G_1(\mathbf{k}_+) G_2(\mathbf{k}_-) \delta(\tilde{\mathbf{k}}_+ - \mathbf{k}_+), \quad (\text{A6.49})$$

where G_0 represents a suitable sum over connected graphs with four external lines, while G_1 and G_2 represent suitable sums over connected graphs with two external lines.

Using the symmetry of the propagator $D_\omega(\mathbf{k}) = i\omega D_\omega(\mathbf{k}^*)$, where, if $\mathbf{k} = (k_0, \mathbf{k})$, $\mathbf{k}^* = (k, -\mathbf{k})$, one easily gets the following symmetry properties for the functions appearing in (9.48):

$$G_{+, \omega}^{(-1)}(\tilde{\mathbf{k}}_+, \mathbf{k}_+, \mathbf{k}_-) = -\omega G_{+, \omega}^{(-1)}(\tilde{\mathbf{k}}_+^*, \mathbf{k}_+^*, \mathbf{k}_-^*), \quad \mathbf{p} \cdot \mathbf{S}_+^{(0)}(\mathbf{k}_+, \mathbf{k}_-) = -i\mathbf{p}^* \cdot \mathbf{S}_+^{(0)}(\mathbf{k}_+^*, \mathbf{k}_-^*). \quad (\text{A6.50})$$

The last equation implies that

$$\hat{F}_{2,+, \omega}^{(-1)}(\mathbf{k}_+, \mathbf{k}_-) = \frac{1}{D_+(\mathbf{p})} [p_0 \hat{A}_{+, \omega, 0}^{(-1)}(\mathbf{k}_+, \mathbf{k}_-) + p \hat{A}_{+, \omega, 1}^{(-1)}(\mathbf{k}_+, \mathbf{k}_-)], \quad (\text{A6.51})$$

where $\hat{A}_{+, \omega, i}^{(-1)}$ are smooth functions satisfying

$$\hat{A}_{+, \omega, 1}^{(-1)}(\mathbf{k}_+, \mathbf{k}_-) = i\omega \hat{A}_{+, \omega, 1}^{(-1)}(\mathbf{k}_+^*, \mathbf{k}_-^*). \quad (\text{A6.52})$$

It follows that, if we define

$$\mathcal{L}\hat{F}_{2,+, \omega}^{(-1)}(\mathbf{k}_+, \mathbf{k}_-) = \frac{1}{D_+(\mathbf{p})} [p_0 \hat{A}_{+, \omega, 0}^{(-1)}(\mathbf{0}, \mathbf{0}) + p \hat{A}_{+, \omega, 1}^{(-1)}(\mathbf{0}, \mathbf{0})], \quad (\text{A6.53})$$

then

$$\mathcal{L}\hat{F}_{2,+,+}^{(-1)}(\mathbf{k}_+, \mathbf{k}_-) = \nu_{-1}^+, \quad \mathcal{L}\hat{F}_{2,+,-}^{(-1)}(\mathbf{k}_+, \mathbf{k}_-) = \nu_{-1}^- \frac{D_-(\mathbf{p})}{D_+(\mathbf{p})}, \quad (\text{A6.54})$$

where ν_{-1}^\pm are real constants, as it can be verified by symmetry.

We now consider the contribution $F_{1,+}^{(-1)}$ in (A6.47). Its Fourier transform has two contributions, the first of the form

$$\hat{F}_{1,+}^{(-1)}(\mathbf{k}_+, \mathbf{k}_-) = \frac{[C_{h,0}(\mathbf{k}_-) - 1]D_+(\mathbf{k}_-)\hat{g}_+^{(0)}(\mathbf{k}_+) - u_0(\mathbf{k}_+)}{D_+(\mathbf{p})}G_+^{(2)}(\mathbf{k}_+), \quad (\text{A6.55})$$

where u_0 is defined in Appendix A7, see (A7.4), and the second possible contribution has the same form of (A6.55), with \mathbf{k}_+ and \mathbf{k}_- interchanged. The natural way to regularize (A6.55) is to define

$$\mathcal{L}\hat{F}_{1,+}^{(-1)}(\mathbf{k}_+, \mathbf{k}_-) = \frac{[C_{h,0}(\mathbf{k}_-) - 1]D_+(\mathbf{k}_-)\hat{g}_+^{(0)}(\mathbf{k}_+) - u_0(\mathbf{k}_+)}{D_+(\mathbf{p})}G_+^{(2)}(\mathbf{0}). \quad (\text{A6.56})$$

Note however that $G_\omega^{(2)}(\mathbf{0}) = 0$, by parity, so the local part in (A6.56) is vanishing. In other words the dimensional gain here is obtained without the introduction of a renormalization constant. The same procedure can be defined for the term obtained by interchanging \mathbf{k}_+ and \mathbf{k}_- in (A6.55).

We can summarize the previous discussion by defining

$$\mathcal{L}K_J^{(-1,2)}(\psi) = \frac{1}{Z_{-1}} \int d\mathbf{x} \left\{ J_{\mathbf{x},+} \left[T_{\mathbf{x}} + \nu_{-1}^+ \psi_{\mathbf{x},+}^+ \psi_{\mathbf{x},+}^- \right] + \nu_{-1}^- J_{\mathbf{x},+}^{(-)} \psi_{\mathbf{x},-}^+ \psi_{\mathbf{x},-}^- \right\} \quad (\text{A6.57})$$

where, if $\hat{J}_{\mathbf{p},+}$ is the Fourier transform of $J_{\mathbf{x},+}$, $J_{\mathbf{x},+}^{(-)}$ is the Fourier transform of $\hat{J}_{\mathbf{p},+} D_-(\mathbf{p})/D_+(\mathbf{p})$.

We are now ready to describe the general step, by defining the action of \mathcal{L} over $K_J^{j,2}$, which can be written, if $j < 1$, after rescaling $\psi^{[h,j]}$, as

$$\begin{aligned} K_J^{(j,2)}(\psi) = & \frac{1}{Z_j} \int d\mathbf{x} \left\{ J_{\mathbf{x},+} T_{\mathbf{x}} + \sum_{\omega} \int d\mathbf{y} d\mathbf{z} \left[J_{\mathbf{x},+} F_{\nu^+,+,\omega}^{(j)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) + J_{\mathbf{x},+}^{(-)} F_{\nu^-,+,\omega}^{(j)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) + \right. \right. \\ & \left. \left. + J_{\mathbf{x},+} F_{2,+, \omega}^{(j)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) + \delta_{+, \omega} J_{\mathbf{x},+} F_{1,\omega}^{(j)}(\mathbf{x}, \mathbf{y}, \mathbf{z}) \right] \psi_{\mathbf{y},\omega}^+ \psi_{\mathbf{z},\omega}^- \right\} \end{aligned} \quad (\text{A6.58})$$

where $F_{\nu^\pm,+, \omega}^{(j)}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ represent the sum over all graphs with one vertex of type ν^\pm and two ψ external fields of type ω , $F_{2,+, \omega}^{(j)}$ is the sum over the same kind of graphs with one vertex $T_{\mathbf{x}}$, whose ψ fields are both contracted and $F_{1,\omega}^{(j)}$ is the sum over the graphs with one vertex $T_{\mathbf{x}}$, such that one of its fields is external.

It is important to stress that, thanks to the support properties of $C_\omega(\mathbf{k}_+, \mathbf{k}_-)$, given a graph contributing to $F_{2,+, \omega}^{(j)}$, at least one of the ψ fields belonging to $T_{\mathbf{x}}$ is contracted on scale 0. This property will give crucial dimensional gains (through the short memory property) for the contributions to the Beta function for ν^\pm coming from $F_{2,+, \omega}^{(j)}$. It is clear that, because of this property, $F_{2,+, \omega}^{(j)}$ can be rewritten as

$$\hat{F}_{2,+, \omega}^{(j)}(\mathbf{k}_+, \mathbf{k}_-) = \frac{\mathbf{p}}{D_+(\mathbf{p})} \sum_{i=j}^0 \int d\tilde{\mathbf{k}}_+ \tilde{\mathbf{S}}_+^{(j)}(\tilde{\mathbf{k}}_+, \tilde{\mathbf{k}}_+ - \mathbf{p}) G_{+, \omega}^{(j)}(\tilde{\mathbf{k}}_+, \mathbf{k}_+, \mathbf{k}_-), \quad (\text{A6.59})$$

for suitable functions $\tilde{\mathbf{S}}_+^{(j)}$ and $G_{+, \omega}^{(j)}$ satisfying the same symmetry properties of (A6.50). Then, the action of \mathcal{L} over $\hat{F}_{2,+, \omega}^{(j)}(\mathbf{k}_+, \mathbf{k}_-)$ is defined exactly as for $j = -1$. Moreover we define

$$\mathcal{L}\hat{F}_{\nu^\pm,+, \omega}^{(j)}(\mathbf{k}_+, \mathbf{k}_-) = \hat{F}_{\nu^\pm,+, \omega}^{(j)}(\mathbf{0}, \mathbf{0}), \quad (\text{A6.60})$$

and, finally, we note that, for the same reasons as in the $j = -1$ case

$$\hat{F}_{\nu^+,+,-}^{(j)}(\mathbf{0}, \mathbf{0}) = \hat{F}_{\nu^-,+,+}^{(j)}(\mathbf{0}, \mathbf{0}) = \hat{F}_{1,+}^{(j)}(\mathbf{0}, \mathbf{0}) = 0, \quad (\text{A6.61})$$

so that the corresponding kernels are automatically regularized, without need of defining any non trivial action of \mathcal{L} .

It follows that we can write

$$\mathcal{L}K_j^{(j,2)}(\psi) = \frac{1}{Z_j} \int d\mathbf{x} \left\{ J_{\mathbf{x},+} \left[T_{\mathbf{x}} + \nu_j^+ \psi_{\mathbf{x},+}^+ \psi_{\mathbf{x},+}^- \right] + \nu_j^- J_{\mathbf{x},+}^{(-)} \psi_{\mathbf{x},-}^+ \psi_{\mathbf{x},-}^- \right\} \quad (\text{A6.62})$$

which defines the renormalization constants ν_j^\pm . Note that ν_j^\pm is built by contribution that either contain another constant ν_k^\pm , $k > j$, or contain a T vertex, which is on scale 0, in the sense explained before (A6.59).

After writing $\mathcal{V}^{(j)} = \mathcal{L}\mathcal{V}^{(j)} + \mathcal{R}\mathcal{V}^{(j)}$ and $K^{(j)} = \mathcal{L}K^{(j)} + \mathcal{R}K^{(j)}$, the next step is to *renormalize* the free measure $P_{\tilde{Z}_j, C_{h,j}}(d\psi^{[h,j]})$, by adding to it part of $\mathcal{L}\mathcal{V}^{(j)}$. We get

$$\begin{aligned} \int P_{\tilde{Z}_j, C_{h,j}}(d\psi^{[h,j]}) e^{-\mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) + K^{(j)}(\sqrt{Z_j}\psi^{[h,j]})} = \\ = e^{-M^2 t_j} \int P_{\tilde{Z}_{j-1}, C_{h,j}}(d\psi^{[h,j]}) e^{-\tilde{\mathcal{V}}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) + \tilde{K}^{(j)}(\sqrt{Z_j}\psi^{[h,j]})}, \end{aligned} \quad (\text{A6.63})$$

where

$$\begin{aligned} \tilde{Z}_{j-1}(\mathbf{k}) &= Z_j[1 + \chi_{h,j}(\mathbf{k})z_j], \\ \tilde{\mathcal{V}}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) &= \mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) - z_j Z_j F_\zeta^{[h,j]}, \end{aligned} \quad (\text{A6.64})$$

where $F_\zeta^{[h,j]}$ is defined as in (5.10) and the factor $\exp(-M^2 t_j)$ in (A6.63) takes into account the different normalization of the two measures. Moreover

$$\tilde{K}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) = \tilde{\mathcal{B}}_\phi^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) + K_J^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) + W_R^{(j)}, \quad (\text{A6.65})$$

where $\tilde{\mathcal{B}}_\phi^{(j)}$ is obtained from $\mathcal{B}_\phi^{(j)}$ by inserting (A6.64) in the second line of (A6.44) and by absorbing the terms proportional to z_j in the terms in the third line of (A6.44).

If $j > h$, the r.h.s of (A6.63) can be written as

$$e^{-M^2 t_j} \int P_{\tilde{Z}_{j-1}, C_{h,j-1}}(d\psi^{[h,j-1]}) \int P_{Z_{j-1}, \tilde{f}_j^{-1}}(d\psi^{(j)}) e^{-\tilde{\mathcal{V}}^{(j)}(\sqrt{Z_j}[\psi^{[h,j-1]} + \psi^{(j)}]) + \tilde{K}^{(j)}(\sqrt{Z_j}[\psi^{[h,j-1]} + \psi^{(j)}])}, \quad (\text{A6.66})$$

where $P_{Z_{j-1}, \tilde{f}_j^{-1}}(d\psi^{(j)})$ is the integration with propagator $\hat{g}_\omega^{(j)}(\mathbf{k})$.

We now *rescale* the field so that

$$\tilde{\mathcal{V}}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) = \hat{\mathcal{V}}^{(j)}(\sqrt{Z_{j-1}}\psi^{[h,j]}) \quad , \quad \tilde{K}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) = \hat{K}^{(j)}(\sqrt{Z_{j-1}}\psi^{[h,j]}) ; \quad (\text{A6.67})$$

it follows that

$$\mathcal{L}\hat{\mathcal{V}}^{(j)}(\psi^{[h,j]}) = \lambda_j F_\lambda^{[h,j]}, \quad (\text{A6.68})$$

where $\lambda_j = (Z_j Z_{j-1}^{-1})^2 l_j$. If we now define

$$\begin{aligned} e^{-\mathcal{V}^{(j-1)}(\sqrt{Z_{j-1}}\psi^{[h,j-1]}) + K^{(j-1)}(\sqrt{Z_{j-1}}\psi^{[h,j-1]}) - M^2 E_j} = \\ = \int P_{Z_{j-1}, \tilde{f}_j^{-1}}(d\psi^{(j)}) e^{-\hat{\mathcal{V}}^{(j)}(\sqrt{Z_{j-1}}[\psi^{[h,j-1]} + \psi^{(j)}]) + \hat{K}^{(j)}(\sqrt{Z_{j-1}}[\psi^{[h,j-1]} + \psi^{(j)}])}, \end{aligned} \quad (\text{A6.69})$$

it is easy to see that $\mathcal{V}^{(j-1)}$ and $K^{(j-1)}$ are of the same form of $\mathcal{V}^{(j)}$ and $K^{(j)}$ and that the procedure can be iterated. Note that the above procedure allows, in particular, to write the running coupling constant λ_{j-1} , $0 > j-1 \geq h$, in terms of $\lambda_{j'}$, $0 \geq j' \geq j$:

$$\lambda_{j-1} = \lambda_j + \beta_\lambda^j(\lambda_j, \dots, \lambda_0) \quad , \quad \lambda_0 = \lambda, \quad (\text{A6.70})$$

and the renormalization constants ν_{j-1}^\pm in terms of λ_k, ν_k^\pm , $k \geq j$:

$$\nu_{j-1}^\alpha = \nu_j^\alpha + \beta_j^{\nu, \alpha}(\lambda_j, \nu_j^\pm; \dots; \lambda_0, \nu_0^\pm) \quad , \quad \nu_0^\alpha = \nu_\alpha \quad , \quad \alpha = \pm. \quad (\text{A6.71})$$

The functions $\beta_j^\lambda(\lambda_{j+1}, \dots, \lambda_0)$ and $\beta_j^{\nu, \alpha}(\lambda_j, \nu_j^\pm; \dots; \lambda_0, \nu_0^\pm)$ are called the λ and the ν^α component of the *Beta function*, respectively. Both functions can be represented by a tree expansion similar to that exposed in Chapter 5, and we do not repeat here the details.

We now want to show that, if $\bar{\lambda}_h \leq \varepsilon_0$ is small enough, it is possible to choose ν^\pm as suitable functions of λ , in such a way that $|\nu_j^\pm| \leq C\varepsilon_0\gamma^{\vartheta j}$, for some $\vartheta > 0$. If we manage to prove this, it will soon follow that $\hat{H}_+^{2,1}$, when computed on the IR cutoff scale, can be bounded by (A6.35), that is by the dimensional bound for $\hat{G}^{2,1}$ times an exponentially small factor $\gamma^{\vartheta h}$. In fact the renormalized expansion for $\hat{H}_+^{2,1}$ contains contributions that either contain a T_x vertex on scale 0, or a ν_j^\pm , with $h < j \leq 0$. If $|\nu_j^\pm| \leq C\varepsilon_0\gamma^{\vartheta j}$, using the short memory property, it is immediate to verify that both contributions are exponentially small w.r.t. the dimensional bound for $\hat{G}^{2,1}$.

So, let us prove the bound on the renormalization constants ν_j^\pm . We rewrite $\beta_j^{\nu, \alpha}$ by distinguishing the contributions independent of ν_k^\pm (which necessarily contain a T vertex on scale 0) and the contribution linear in ν_k^\pm , $k \geq j$:

$$\beta_j^{\nu, \alpha}(\lambda_j, \nu_j^\pm; \dots; \lambda_0, \nu_0^\pm) = \beta_{j,1}^{\nu, \alpha}(\lambda_j, \dots, \lambda_0) + \sum_{k=j}^0 \sum_{\omega=\pm} \nu_k^\omega \beta_{j,k}^{\nu, \alpha, \omega}(\lambda_j, \dots, \lambda_0). \quad (\text{A6.72})$$

Moreover, by the short memory property, there exists $0 < \vartheta < 1/4$ and positive constants c_1 and c_2 such that

$$|\beta_{j,1}^{\nu, \alpha}(\lambda_j, \dots, \lambda_0)| \leq c_1 \bar{\lambda}_h \gamma^{2\vartheta j} \quad , \quad |\beta_{j,k}^{\nu, \alpha, \omega}(\lambda_j, \dots, \lambda_0)| \leq c_2 \bar{\lambda}_h^2 \gamma^{2\vartheta(j-j')}. \quad (\text{A6.73})$$

By iterating (A6.71), we find:

$$\nu_{j-1}^\alpha = \nu_0^\alpha + \sum_{k=j}^0 \beta_k^{\nu, \alpha}(\lambda_k, \nu_k^\pm; \dots; \lambda_0, \nu_0^\pm), \quad (\text{A6.74})$$

so that, imposing the condition $\nu_h^\pm \equiv 0$, we get:

$$\nu_0^\alpha = - \sum_{k=h+1}^0 \beta_k^{\nu, \alpha}(\lambda_k, \nu_k^\pm; \dots; \lambda_0, \nu_0^\pm). \quad (\text{A6.75})$$

Inserting the last equation into (A6.74) we get:

$$\nu_j^\alpha = - \sum_{k=h+1}^j \beta_k^{\nu, \alpha}(\lambda_k, \nu_k^\pm; \dots; \lambda_0, \nu_0^\pm). \quad (\text{A6.76})$$

In other words, the condition $\nu_h^\pm \equiv 0$ can be satisfied iff it can be found a sequence $\underline{\nu} = \{\nu_j^\omega\}_{h \leq j \leq 0}^{\omega=\pm}$ satisfying (A6.76). In order to prove that this is possible, we introduce the space \mathfrak{M}_ϑ of the sequences

$\underline{\nu}$ such that $\max_{\omega} |\nu_j^{\omega}| \leq c \bar{\lambda}_h \gamma^{\vartheta j}$, for some c ; we shall think \mathfrak{M}_{ϑ} as a Banach space with norm $\|\underline{\nu}\|_{\vartheta} = \sup_{h+1 \leq j \leq 0} \max_{\omega} |\nu_j^{\omega}| \gamma^{-\vartheta j} \bar{\lambda}_h^{-1}$. We then look for a fixed point of the operator $\mathbf{T} : \mathfrak{M}_{\vartheta} \rightarrow \mathfrak{M}_{\vartheta}$ defined as:

$$(\mathbf{T}\underline{\nu})_j^{\alpha} = - \sum_{k=h+1}^j \beta_k^{\nu, \alpha}(\lambda_k, \nu_k^{\pm}; \dots; \lambda_0, \nu_0^{\pm}) . \quad (\text{A6.77})$$

Note that, if $\bar{\lambda}_h$ is sufficiently small, then \mathbf{T} leaves invariant the ball \mathfrak{B}_{ϑ} of radius $c_0 = 2c_1 \sum_{n=0}^{\infty} \gamma^{-\vartheta n}$ of \mathfrak{M}_{ϑ} , c_1 being the constant in (A6.73). In fact, by (A6.72) and (A6.73), if $\|\underline{\nu}\|_{\vartheta} \leq c_0$, then

$$|(\mathbf{T}\underline{\nu})_j^{\alpha}| \leq \sum_{k=h+1}^j c_1 \bar{\lambda}_h \gamma^{2\vartheta k} + 2 \sum_{k=h+1}^j \sum_{i=k}^0 c_0 \bar{\lambda}_h \gamma^{\vartheta i} c_2 \bar{\lambda}_h^2 \gamma^{2\vartheta(k-i)} \leq c_0 \bar{\lambda}_h \gamma^{\vartheta j} , \quad (\text{A6.78})$$

if $4c_2 \bar{\lambda}_h^2 (\sum_{n=0}^{\infty} \gamma^{-\vartheta n})^3 \leq 1$.

\mathbf{T} is also a contraction on \mathfrak{B}_{ϑ} , if $\bar{\lambda}_h$ is sufficiently small; in fact, if $\underline{\nu}, \underline{\nu}' \in \mathfrak{M}_{\vartheta}$,

$$\begin{aligned} |(\mathbf{T}\underline{\nu})_j^{\alpha} - (\mathbf{T}\underline{\nu}')_j^{\alpha}| &\leq \sum_{k=h+1}^j |\beta_k^{\nu, \alpha}(\lambda_k, \nu_k^{\pm}; \dots; \lambda_0, \nu_0^{\pm}) - \beta_k^{\nu', \alpha}(\lambda_k, \nu'_k{}^{\pm}; \dots; \lambda_0, \nu'_0{}^{\pm})| \\ &\leq 2 \sum_{k=h+1}^j \sum_{i=k}^0 \|\underline{\nu} - \underline{\nu}'\|_{\vartheta} \bar{\lambda}_h \gamma^{\vartheta i} c_2 \bar{\lambda}_h^2 \gamma^{2\vartheta(k-i)} \leq \frac{1}{2} \|\underline{\nu} - \underline{\nu}'\|_{\vartheta} \bar{\lambda}_h \gamma^{\vartheta j} , \end{aligned} \quad (\text{A6.79})$$

if $4c_2 \bar{\lambda}_h^2 (\sum_{n=0}^{\infty} \gamma^{-\vartheta n})^3 \leq 1$, as above. Hence, by the contraction principle, there is a unique fixed point $\underline{\nu}^*$ of \mathbf{T} on \mathfrak{B}_{ϑ} . This concludes the proof of the exponential decay of ν_j^{\pm} and, as discussed above, of the first correction identity. \blacksquare

A6.5. Ward identities and the second addend in (A6.8)

Starting from the present section, we begin to deal with the second term in the r.h.s. of (A6.8), with the aim of showing that it admits a good dimensional bound, as discussed for the first one. The vanishing of the λ component of the Beta function will be an easy consequence of such a good dimensional bound.

The strategy will be the following: in the present section we will first describe two more Ward identities connecting $\hat{G}_+^{4,1}$ with \hat{G}^4 . As the first Ward identity considered above, they will have a correction due to the cutoff function, and these corrections will satisfy new correction identities, presented below in this section. The proof of the new correction identities, which is the main difficulty of all the proof of the present Appendix, will be presented in next section. The present section will be concluded with the proof of the vanishing of the Beta function, obtained by a careful use of the new Ward identities together with the new correction identities.

The new pair of Ward identities we need here is the following, see Fig. 6.

$$\begin{aligned} D_+(\mathbf{p}) \hat{G}_+^{4,1}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) &= \hat{G}_+^4(\mathbf{k}_1 - \mathbf{p}, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) - \hat{G}_+^4(\mathbf{k}_1, \mathbf{k}_2 + \mathbf{p}, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) + \hat{\Delta}_+^{4,1} , \\ D_-(\mathbf{p}) \hat{G}_-^{4,1}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) &= \hat{G}_+^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 - \mathbf{p}, \mathbf{k}_4 - \mathbf{p}) - \hat{G}_+^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) + \hat{\Delta}_-^{4,1} , \end{aligned} \quad (\text{A6.80})$$

where $\hat{\Delta}_{\pm}^{4,1}$ are the “correction terms”

$$\hat{\Delta}_{\pm}^{4,1}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{1}{M^2} \sum_{\mathbf{k}} C_{\pm}(\mathbf{k}, \mathbf{k} - \mathbf{p}) < \hat{\psi}_{\mathbf{k}, \pm}^+ \hat{\psi}_{\mathbf{k} - \mathbf{p}, \pm}^-; \hat{\psi}_{\mathbf{k}_1, +}^-; \hat{\psi}_{\mathbf{k}_2, +}^+; \hat{\psi}_{\mathbf{k}_3, -}^-; \hat{\psi}_{\mathbf{k}_4 - \mathbf{p}, -}^+ >^T . \quad (\text{A6.81})$$

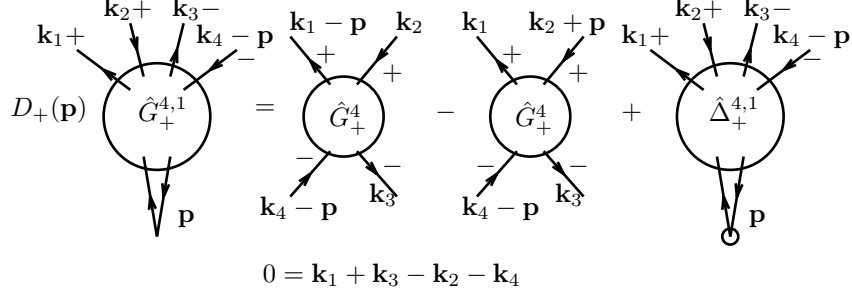


FIG. 6. Graphical representation of the Ward identity (A6.19)

The Ward identities (A6.80) can be derived from (A6.23) by deriving four times w.r.t. the external ϕ fields. By adding and subtracting suitable counterterms¹ ν_{\pm} , the first of (A6.80) can be rewritten as

$$(1 - \nu_+)D_+(\mathbf{p})\hat{G}_+^{4,1}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) - \nu_-D_-(\mathbf{p})\hat{G}_-^{4,1}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) \\ = \hat{G}_+^4(\mathbf{k}_1 - \mathbf{p}, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) - \hat{G}_+^4(\mathbf{k}_1, \mathbf{k}_2 + \mathbf{p}, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) + \hat{H}_+^{4,1}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}), \quad (\text{A6.82})$$

where by definition

$$\hat{H}_+^{4,1}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) = \frac{1}{M^2} \sum_{\mathbf{k}} C_+(\mathbf{k}, \mathbf{k} - \mathbf{p}) < \hat{\psi}_{\mathbf{k},+}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,-}^-; \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ >^T - \\ - \frac{1}{M^2} \sum_{\mathbf{k}} \sum_{\omega} \nu_{\omega} D_{\omega}(\mathbf{p}) < \hat{\psi}_{\mathbf{k},\omega}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},\omega}^-; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,-}^-; \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ >^T. \quad (\text{A6.83})$$

In the same way, in terms of new counterterms ν'_{\pm} , the second of (A6.80) can be written as

$$(1 - \nu'_-)D_-(\mathbf{p})\hat{G}_-^{4,1}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) - \nu'_+D_+(\mathbf{p})\hat{G}_+^{4,1}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) = \\ = \hat{G}_+^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 - \mathbf{p}, \mathbf{k}_4 - \mathbf{p}) - \hat{G}_+^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) + \hat{H}_-^{4,1}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}), \quad (\text{A6.84})$$

where

$$\hat{H}_-^{4,1}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) = \frac{1}{M^2} \sum_{\mathbf{k}} C_-(\mathbf{k}, \mathbf{k} - \mathbf{p}) < \hat{\psi}_{\mathbf{k},-}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},-}^-; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,-}^-; \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ >^T - \\ - \frac{1}{M^2} \sum_{\mathbf{k}} \sum_{\omega} \nu'_{\omega} D_{\omega}(\mathbf{p}) < \hat{\psi}_{\mathbf{k},\omega}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},\omega}^-; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,-}^-; \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ >^T. \quad (\text{A6.85})$$

If we insert in the r.h.s. of (A6.82) the value of $\hat{G}_-^{4,1}$ taken from (A6.84), we get

$$(1 + A)D_+(\mathbf{p})\hat{G}_+^{4,1}(\mathbf{p}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) = \hat{G}_+^4(\mathbf{k}_1 - \mathbf{p}, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) - \\ - \hat{G}_+^4(\mathbf{k}_1, \mathbf{k}_2 + \mathbf{p}, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) + B \left[\hat{G}_+^4(\mathbf{k}_1, \mathbf{k}_3 - \mathbf{p}, \mathbf{k}_4 - \mathbf{p}) - \hat{G}_+^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \right] + \hat{H}_+^{4,1} + B\hat{H}_-^{4,1}, \quad (\text{A6.86})$$

¹ with an abuse of notation, here we call the counterterms with the same symbols as those used for the first correction identity; note that here the new counterterms are different from those of previous section.

where

$$A = -\nu_+ - \frac{\nu_- \nu'_+}{1 - \nu'_-} \quad , \quad B = \frac{\nu_-}{1 - \nu'_-} . \quad (\text{A6.87})$$

Let us now consider the second term in the r.h.s. of (A6.8) and let us rewrite it as:

$$\lambda \hat{g}_-(\mathbf{k}_4) \left[\frac{1}{M^2} \sum_{\mathbf{p}} \chi_M(\mathbf{p}) \hat{G}_+^{4,1}(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) + \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \hat{G}_+^{4,1}(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) \right] , \quad (\text{A6.88})$$

where: $\chi_M(\mathbf{p})$ is a cutoff function vanishing for scales bigger than $h + \log_\gamma 2$ (*i.e.* the presence of $\chi_M(\mathbf{p})$ constraints the transferred momentum to be $\leq O(\gamma^h)$); $\tilde{\chi}_M(\mathbf{p})$ is a cutoff function vanishing for scales smaller than $h + \log_\gamma 2$ and bigger than $\log_\gamma 2$ (*i.e.* the presence of $\tilde{\chi}_M(\mathbf{p})$ constraints the transferred momentum to be $O(\gamma^h) \leq |\mathbf{p}| \leq O(1)$). The two functions are chosen so that they sum up to 1 in the scales range between h and 0.

If we insert in the last term of (A6.85) the value of $\hat{G}_+^{4,1}$ taken from (A6.86), we get

$$\begin{aligned} & \lambda \hat{g}_-(\mathbf{k}_4) \frac{1}{M^2} \sum_{\mathbf{p}} \chi_M(\mathbf{p}) G_+^{4,1}(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) + \\ & + \frac{\lambda \hat{g}_-(\mathbf{k}_4)}{(1+A)} \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{\hat{G}_+^4(\mathbf{k}_1 - \mathbf{p}, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) - \hat{G}_+^4(\mathbf{k}_1, \mathbf{k}_2 + \mathbf{p}, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p})}{D_+(\mathbf{p})} + \\ & + \frac{\lambda \hat{g}_-(\mathbf{k}_4)}{(1+A)} \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{\hat{G}_+^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 - \mathbf{p}, \mathbf{k}_4 - \mathbf{p}) - \hat{G}_+^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)}{D_+(\mathbf{p})} + \\ & + \frac{\lambda \hat{g}_-(\mathbf{k}_4)}{(1+A)} \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{\hat{H}_+^{4,1}(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) + B \hat{H}_-^{4,1}(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p})}{D_+(\mathbf{p})} . \end{aligned} \quad (\text{A6.89})$$

Note that

$$\frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{\hat{G}_+^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)}{D_+(\mathbf{p})} = 0 , \quad (\text{A6.90})$$

since $D_+(\mathbf{p})$ is odd. Then, (A6.89) computed with the momenta equal to

$$\mathbf{k}_i = \bar{\mathbf{k}}_i \quad , \quad \bar{\mathbf{k}}_1 = \bar{\mathbf{k}}_4 = -\bar{\mathbf{k}}_2 = -\bar{\mathbf{k}}_3 = \bar{\mathbf{k}} \quad , \quad |\bar{\mathbf{k}}| = \gamma^h , \quad (\text{A6.91})$$

is equivalent to

$$\begin{aligned} & \lambda \hat{g}_-(\bar{\mathbf{k}}_4) \frac{1}{M^2} \sum_{\mathbf{p}} \chi_M(\mathbf{p}) G_+^{4,1}(\mathbf{p}; \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4 - \mathbf{p}) + \\ & + \frac{\lambda \hat{g}_-(\bar{\mathbf{k}}_4)}{(1+A)} \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{\hat{G}_+^4(\bar{\mathbf{k}}_1 - \mathbf{p}, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4 - \mathbf{p}) - \hat{G}_+^4(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2 + \mathbf{p}, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4 - \mathbf{p})}{D_+(\mathbf{p})} + \\ & + \frac{\lambda \hat{g}_-(\bar{\mathbf{k}}_4)}{(1+A)} \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{\hat{G}_+^4(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3 - \mathbf{p}, \bar{\mathbf{k}}_4 - \mathbf{p})}{D_+(\mathbf{p})} + \\ & + \frac{\lambda \hat{g}_-(\bar{\mathbf{k}}_4)}{(1+A)} \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{\hat{H}_+^{4,1}(\mathbf{p}; \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4 - \mathbf{p}) + B \hat{H}_-^{4,1}(\mathbf{p}; \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4 - \mathbf{p})}{D_+(\mathbf{p})} . \end{aligned} \quad (\text{A6.92})$$

All the terms appearing in the above equation can be expressed in terms of convergent tree expansions, via a recursive expansion similar to that described in the last section. Dimensional bounds for the terms in

the first three lines can be easily derived, in analogy with the results of Theorem A6.1. In Appendix A1 of [BM2] the following bound is proven:

$$\left| \lambda \hat{g}_-(\bar{\mathbf{k}}_4) \frac{1}{M^2} \sum_{\mathbf{p}} \chi_M(\mathbf{p}) G_+^{4,1}(\mathbf{p}; \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4 - \mathbf{p}) \right| \leq C \bar{\lambda}_h^3 \gamma^{\eta_{2,1}h} \frac{\gamma^{-4h}}{Z_h^2} \quad (\text{A6.93})$$

where the exponent $\eta_{2,1}$ is the same of (A6.33). However, from the result of previous section (*i.e.* from the validity of the first correction identity) it follows that $\eta_{2,1} = 0$, so that the r.h.s. of (A6.93) is the right dimensional bound we need. As regarding the terms in the second and the third line of (A6.89), following a procedure similar to that leading to the second of (A6.34) (see again Appendix A1 of [BM2]), we find

$$\begin{aligned} & \left| \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{\hat{G}_+^4(\bar{\mathbf{k}}_1 - \mathbf{p}, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4 - \mathbf{p}) - \hat{G}_+^4(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2 + \mathbf{p}, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4 - \mathbf{p})}{D_+(\mathbf{p})} + \right. \\ & \left. + \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{\hat{G}_+^4(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3 - \mathbf{p}, \bar{\mathbf{k}}_4 - \mathbf{p})}{D_+(\mathbf{p})} \right| \leq C \bar{\lambda}_h \frac{\gamma^{-3h}}{Z_h^2}, \end{aligned} \quad (\text{A6.94})$$

that, again, is the right dimensional bound.

The bound on the term in the last line of (A6.89) is more involved, and require an analysis similar (but more complicated) to that of previous section. We will prove in next section that

there exists $\varepsilon_1 \leq \varepsilon_0$ and four λ -functions $\nu_+, \nu_-, \nu'_+, \nu'_-$ of order λ (uniformly in h), such that, if $\bar{\lambda}_h \leq \varepsilon_1$,

$$\left| \lambda \hat{g}_-(\bar{\mathbf{k}}_4) \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{\hat{H}_{\pm}^{4,1}(\mathbf{p}; \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4 - \mathbf{p})}{D_+(\mathbf{p})} \right| \leq C \bar{\lambda}_h^2 \frac{\gamma^{-4h}}{Z_h^2} \quad (\text{A6.95})$$

Substituting the bounds (A6.95), (A6.93), (A6.94) together with (A6.37) and the second of (A6.34) in the Dyson equation (A6.8), we finally get

$$|\lambda_h| \leq c|\lambda|(1 + O(\bar{\lambda}_h)), \quad (\text{A6.96})$$

which implies

THEOREM A6.2 *The model (A6.3) is well defined in the limit $h \rightarrow -\infty$. In fact there are constants ε_1 and c_2 such that $|\lambda| \leq \varepsilon_1$ implies $\bar{\lambda}_j \leq c_2 \varepsilon_1$, for any $j < 0$.*

Finally, a standard argument shows that, as a consequence of Theorem A6.2, the first bound in (6.6) holds, that is

$$|\beta_{\lambda}^h(\lambda_h, \dots, \lambda_h)| \leq C |\bar{\lambda}_h|^2 \gamma^{\vartheta h}, \quad (\text{A6.97})$$

The proof is by contradiction. Consider the Taylor expansion of $\beta_{\lambda}^h(\lambda_h, \dots, \lambda_h)$ in λ_h (which is convergent for λ_h small enough) and let us call $b_r^{(h)}$ the coefficient of $(\lambda_h)^r$. Let us also call $b_r \stackrel{\text{def}}{=} \lim_{h \rightarrow -\infty} b_r^{(h)}$. By performing the bounds on the trees representing $b_r^{(h)}$, in the same way explained in Chapter 5, we find that necessarily $b_r^{(h)} = b_r + O(\gamma^{\vartheta h})$, for some $\vartheta > 0$. Now, let us assume by contradiction that, for some $r \geq 2$,

$$\beta_{\lambda}^h(\lambda_h, \dots, \lambda_h) = b_r(\lambda_h)^r + O(|\lambda_h|^{r+1}) + O(\lambda_h^2 \gamma^{\vartheta h}), \quad (\text{A6.98})$$

with b_r a non vanishing constant. By the discussion above and Theorem 5.1, the running coupling constants λ_h are analytic functions of λ :

$$\lambda_h = \lambda + \sum_{n=2}^r c_n^{(h)} \lambda^n + O(\lambda^{r+1}) \quad (\text{A6.99})$$

and for any fixed h the sequence $c_n^{(h)}$ is a bounded sequence.

Consider the flow equation (A6.70) and rewrite it as

$$\lambda_{h-1} = \lambda_h + \beta_\lambda^h(\lambda_h, \dots, \lambda_h) + \sum_{k \geq h} D_\lambda^{h,k}, \quad (\text{A6.100})$$

where

$$D_\lambda^{h,k} = \beta_\lambda^h(\lambda_h, \dots, \lambda_h, \lambda_k, \lambda_{k+1}, \dots, \lambda_1) - \beta_\lambda^h(\lambda_h, \dots, \lambda_h, \lambda_h, \lambda_{k+1}, \dots, \lambda_1). \quad (\text{A6.101})$$

and, by using the short memory property, it is easy to show that $|D_\lambda^{h,k}| \leq c \bar{\lambda}_h \gamma^{\vartheta(h-k)} |\lambda_h - \lambda_k|$, for some $\vartheta > 0$

Inserting (A6.98) and (A6.99) into (A6.100) and keeping at both sides the terms of order r , we find:

$$c_r^{(h-1)} = c_r^{(h)} + b_r + O(\lambda^2 \gamma^{\vartheta h}). \quad (\text{A6.102})$$

This would mean that $c_r^{(h)}$ is a sequence diverging for $h \rightarrow -\infty$, which is in contradiction with the fact, following from Theorem A6.2 and the discussin above, that λ_h is an analytic function of λ , uniformly in h . ■

A6.6.Proof of (A6.95)

In this final section we prove the bound (A6.95), that is we conclude the proof of the second pair of correction identities and, with this, we conclude the proof of Theorem A6.2, that is the main result of this Appendix. We shall prove first the bound (A6.95) for $\hat{H}_+^{4,1}$; the bound for $\hat{H}_-^{4,1}$ is done essentially in the same way and will be briefly discussed later. By using (A6.83), we get

$$\begin{aligned} \hat{g}_-(\mathbf{k}_4) \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) D_+^{-1}(\mathbf{p}) \hat{H}_+^{4,1}(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) = \\ = \hat{g}_-(\mathbf{k}_4) \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{1}{M^2} \sum_{\mathbf{k}} \frac{C_+(\mathbf{k}, \mathbf{k} - \mathbf{p})}{D_+(\mathbf{p})} < \hat{\psi}_{\mathbf{k},+}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,-}^-; \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ >^T - \\ - \nu_- \hat{g}_-(\mathbf{k}_4) \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{1}{M^2} \sum_{\mathbf{k}} \frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} < \hat{\psi}_{\mathbf{k},-}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},-}^-; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,+}^-; \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ >^T - \\ - \nu_+ \hat{g}_-(\mathbf{k}_4) \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{1}{M^2} \sum_{\mathbf{k}} < \hat{\psi}_{\mathbf{k},+}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,-}^-; \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ >^T. \end{aligned} \quad (\text{A6.103})$$

Let us define

$$\tilde{G}_+^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \frac{\partial^4}{\partial \phi_{\mathbf{k}_1,+}^+ \partial \phi_{\mathbf{k}_2,+}^- \partial \phi_{\mathbf{k}_3,-}^+ \partial J_{\mathbf{k}_4}} \tilde{\mathcal{W}} \Big|_{\phi=0}, \quad (\text{A6.104})$$

where

$$\tilde{\mathcal{W}} = \log \int P(d\hat{\psi}) e^{-T_1(\hat{\psi}) + \nu_+ T_+(\hat{\psi}) + \nu_- T_-(\hat{\psi}) - V(\hat{\psi}) + \sum_{\omega} \int d\mathbf{x} [\phi_{\mathbf{x},\omega}^+ \hat{\psi}_{\mathbf{x},\omega}^- + \hat{\psi}_{\mathbf{x},\omega}^+ \phi_{\mathbf{x},\omega}^-]}, \quad (\text{A6.105})$$

and

$$\begin{aligned} T_1(\psi) &= \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{1}{M^2} \sum_{\mathbf{k}} \frac{C_+(\mathbf{k}, \mathbf{k} - \mathbf{p})}{D_+(\mathbf{p})} (\hat{\psi}_{\mathbf{k},+}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-) \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ \hat{J}_{\mathbf{k}_4} \hat{g}_-(\mathbf{k}_4), \\ T_+(\psi) &= \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{1}{M^2} \sum_{\mathbf{k}} (\hat{\psi}_{\mathbf{k},+}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-) \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ \hat{J}_{\mathbf{k}_4} \hat{g}_-(\mathbf{k}_4), \\ T_-(\psi) &= \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{1}{M^2} \sum_{\mathbf{k}} \frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} (\hat{\psi}_{\mathbf{k},-}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},-}^-) \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ \hat{J}_{\mathbf{k}_4} \hat{g}_-(\mathbf{k}_4). \end{aligned} \quad (\text{A6.106})$$

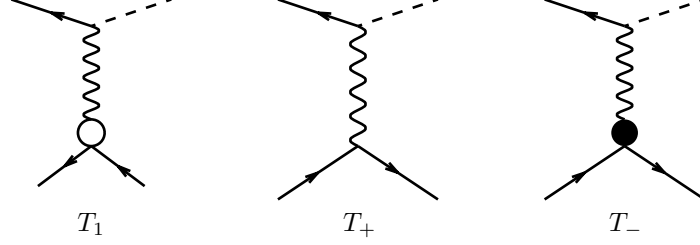


FIG. 7. Graphical representation of T_1, T_+, T_- ; the dotted line carries momentum $\bar{\mathbf{k}}_4$, the empty circle represents C_+ , the filled one $D_-(\mathbf{p})/D_+(\mathbf{p})$

The vertices T_1, T_+ and T_- can be graphically represented as in Fig. 7. The wavy lines represent the cutoff function $\tilde{\chi}_M(\mathbf{p})$, constraining the transferred momentum to be $|\mathbf{p}| \geq O(\gamma^h)$.

It is easy to see that \tilde{G}_+^4 is related to (A6.103) by an identity similar to (A6.13). In fact we can write

$$\begin{aligned}
 & -\tilde{G}_+^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \\
 & = \hat{g}_-(\mathbf{k}_4) \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{1}{M^2} \sum_{\mathbf{k}} \frac{C_+(\mathbf{k}, \mathbf{k} - \mathbf{p})}{D_+(\mathbf{p})} < (\hat{\psi}_{\mathbf{k},+}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-) \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,-}^- >^T - \\
 & - \nu_- \hat{g}_-(\mathbf{k}_4) \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{1}{M^2} \sum_{\mathbf{k}} \frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} < (\hat{\psi}_{\mathbf{k},+}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-) \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,-}^- >^T - \\
 & - \nu_+ \hat{g}_-(\mathbf{k}_4) \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{1}{M^2} \sum_{\mathbf{k}} < (\hat{\psi}_{\mathbf{k},+}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-) \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,-}^- >^T .
 \end{aligned} \tag{A6.107}$$

If we introduce the definitions

$$\delta\rho_{\mathbf{p},+} = \frac{1}{M^2} \sum_{\mathbf{k}} \frac{C_+(\mathbf{p}, \mathbf{k})}{D_+(\mathbf{p})} (\hat{\psi}_{\mathbf{k},+}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-), \quad \rho_{\mathbf{p},+} = \frac{1}{M^2} \sum_{\mathbf{k}} (\hat{\psi}_{\mathbf{k},+}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-), \tag{A6.108}$$

we can rewrite

$$\begin{aligned}
 & \frac{1}{M^2} \sum_{\mathbf{k}} \frac{C_+(\mathbf{k}, \mathbf{k} - \mathbf{p})}{D_+(\mathbf{p})} < (\hat{\psi}_{\mathbf{k},+}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-) \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,-}^- >^T = \\
 & = - < \delta\rho_{\mathbf{p},+}; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,-}^-; \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ >^T - < \delta\rho_{\mathbf{p},+}; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+ >^T < \hat{\psi}_{\mathbf{k}_3,-}^-; \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ >
 \end{aligned} \tag{A6.109}$$

and

$$\begin{aligned}
 & \frac{1}{M^2} \sum_{\mathbf{k}} < (\hat{\psi}_{\mathbf{k},+}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-) \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,-}^- >^T = \\
 & = - < \rho_{\mathbf{p},+}; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,-}^-; \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ >^T - < \rho_{\mathbf{p},+}; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+ >^T < \hat{\psi}_{\mathbf{k}_3,-}^-; \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ >,
 \end{aligned} \tag{A6.110}$$

where we used the fact that $\mathbf{p} \neq 0$ in the support of $\tilde{\chi}_M(\mathbf{p})$ and $< \delta\rho_{\mathbf{p},+} > = < \rho_{\mathbf{p},+} > = 0$ for $\mathbf{p} \neq 0$.

Substituting (A6.109) and (A6.110) into (A6.107), we get

$$\begin{aligned}
 \tilde{G}_+^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) & = g_-(\mathbf{k}_4) \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{H_+^{4,1}(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p})}{D_+(\mathbf{p})} + \\
 & + \tilde{\chi}_M(\mathbf{k}_1 - \mathbf{k}_2) g_-(\mathbf{k}_4) G_-^2(\mathbf{k}_3) \left[< \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \delta\rho_{\mathbf{k}_1-\mathbf{k}_2,+} >^T - \right. \\
 & \left. - \nu_+ < \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \rho_{\mathbf{k}_1-\mathbf{k}_2,+} >^T - \nu_- \frac{D_-(\mathbf{k}_1 - \mathbf{k}_2)}{D_+(\mathbf{k}_1 - \mathbf{k}_2)} < \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \rho_{\mathbf{k}_1-\mathbf{k}_2,-} >^T \right]
 \end{aligned} \tag{A6.111}$$

We now put $\mathbf{k}_i = \bar{\mathbf{k}}_i$, see (A6.91). Since $|\bar{\mathbf{k}}_1 - \bar{\mathbf{k}}_2| = 2\gamma^h$, $\tilde{\chi}_M(\bar{\mathbf{k}}_1 - \bar{\mathbf{k}}_2) = 0$, hence we get

$$\tilde{G}_+^4(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4) = g_-(\bar{\mathbf{k}}_4) \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{H_+^{4,1}(\mathbf{p}; \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4 - \mathbf{p})}{D_+(\mathbf{p})}. \quad (\text{A6.112})$$

Remark. (A6.112) says that the last line of equation (A6.89) can be written as a functional integral very similar to the one for G_+^4 except that the interaction V (A6.2) is replaced by $\mathcal{V} + T_1 - \nu_+ T_+ - \nu_- T_-$; we will evaluate it via a multiscale integration procedure similar to the one for G_+^4 , and in the expansion additional running coupling constants will appear; the expansion is convergent again if such new running couplings will remain small uniformly in the infrared cutoff.

A6.7. The calculation of $\tilde{G}_+^4(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4)$ is done via a multiscale expansion; we shall concentrate on the differences with respect to that described in §A6.4, due to the presence in the potential of the terms $T_1(\psi)$ and $T_{\pm}(\psi)$.

At each step we will write the effective potential $\widetilde{\mathcal{W}}$ as:

$$e^{\widetilde{\mathcal{W}}(\phi, J)} = e^{-M^2 E_j} \int P_{\tilde{Z}_j, C_{h,j}}(d\psi^{[h,j]}) e^{-\mathcal{V}^{(j)}(\sqrt{Z_j} \psi^{[h,j]}) + \mathcal{B}_\phi^{(j)}(\sqrt{Z_j} \psi^{[h,j]}) + K^{(j)}(\psi^{[h,j]}, \phi, J)}, \quad (\text{A6.113})$$

where $\mathcal{V}^{(j)}$ and $\mathcal{B}_\phi^{(j)}$ are defined as in §A6.4, while

$$K^{(j)}(\psi, \phi, J) = \bar{\mathcal{V}}_J^{(j)}(\psi) + W_R^{(j)}(\psi, \phi, J), \quad (\text{A6.114})$$

with $\bar{\mathcal{V}}_J^{(j)}(\psi^{[h,-1]})$ the sum over the terms containing exactly one J field and no ϕ fields and $W_R^{(j)}$ the rest (not involved in the construction of $\tilde{G}_4(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4)$).

The iterative construction of (A6.113) is defined through the analogue of (A6.69):

$$\begin{aligned} e^{-\mathcal{V}^{(j-1)}(\sqrt{Z_{j-1}} \psi^{[h,j-1]}) + \mathcal{B}_\phi(\sqrt{Z_{j-1}} \psi^{[h,j-1]}) + K^{(j-1)}(\psi^{[h,j-1]}, \phi, J) - L\beta E_j} = \\ = \int P_{Z_{j-1}, \tilde{f}_j^{-1}}(d\psi^{(j)}) e^{-\tilde{\mathcal{V}}^{(j)}(\sqrt{Z_{j-1}} [\psi^{[h,j-1]} + \psi^{(j)}]) + \tilde{\mathcal{B}}_\phi(\sqrt{Z_{j-1}} [\psi^{[h,j-1]} + \psi^{(j)}]) + K^{(j)}([\psi^{[h,j-1]} + \psi^{(j)}])}, \end{aligned} \quad (\text{A6.115})$$

Note that in (A6.113) we chose *not to rescale* the fields in $K^{(j)}(\psi, \phi, J)$.

In order to define the action of \mathcal{L} over $\bar{\mathcal{V}}_J^{(j)}(\psi^{[h,-1]})$, let us first consider in detail the first step of the iterative integration procedure, the integration of the field $\psi^{(0)}$. We write

$$\bar{\mathcal{V}}_J^{(-1)}(\psi^{[h,-1]}) = \bar{\mathcal{V}}_{J,a,1}^{(-1)}(\psi^{[h,-1]}) + \bar{\mathcal{V}}_{J,a,2}^{(-1)}(\psi^{[h,-1]}) + \bar{\mathcal{V}}_{J,b,1}^{(-1)}(\psi^{[h,-1]}) + \bar{\mathcal{V}}_{J,b,2}^{(-1)}(\psi^{[h,-1]}), \quad (\text{A6.116})$$

where $\bar{\mathcal{V}}_{J,a,1}^{(-1)} + \bar{\mathcal{V}}_{J,a,2}^{(-1)}$ is the sum of the terms in which the field $\hat{\psi}_{\mathbf{k}_4 - \mathbf{p}, -}^+$ appearing in the definition of $T_1(\psi)$ or $T_{\pm}(\psi)$ is contracted, $\bar{\mathcal{V}}_{J,a,1}^{(-1)}$ and $\bar{\mathcal{V}}_{J,a,2}^{(-1)}$ denoting the sum over the terms of this type containing a T_1 or a T_{\pm} vertex, respectively; $\bar{\mathcal{V}}_{J,b,1}^{(-1)} + \bar{\mathcal{V}}_{J,b,2}^{(-1)}$ is the sum of the other terms, that is those where the field $\hat{\psi}_{\mathbf{k}_4 - \mathbf{p}, -}^+$ is an external field, the index $i = 1, 2$ having the same meaning as before.

Note that the condition (A6.91) on the external momenta \mathbf{k}_i forbids the presence of vertices of type ϕ , if $h < 0$, as we shall suppose. Hence, all graphs contributing to $\bar{\mathcal{V}}_J^{(-1)}$ have, besides the external field of type J , an odd number of external fields of type ψ .

Let us consider first $\bar{\mathcal{V}}_{J,a,1}^{(-1)}$; we shall still distinguish different group of terms, those where both fields $\hat{\psi}_{\mathbf{k}, +}^+$ and $\hat{\psi}_{\mathbf{k} - \mathbf{p}, +}^-$ are contracted, those where only one among them is contracted and those where no one is contracted.

If no one of the fields $\hat{\psi}_{\mathbf{k},+}^+$ and $\hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-$ is contracted, we can only have terms with at least four external lines; for the properties of $\Delta^{(i,j)}$, see Appendix A7, at least one of the fields $\hat{\psi}_{\mathbf{k},+}^+$ and $\hat{\psi}_{\mathbf{k}+\mathbf{p},+}^-$ must be contracted at scale h . If one of these terms has four external lines, hence it is marginal, it has the following form

$$\int d\mathbf{p} \tilde{\chi}_M(\mathbf{p}) \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ G_2^{(0)}(\bar{\mathbf{k}}_4 - \mathbf{p}) \hat{g}_-^{(0)}(\bar{\mathbf{k}}_4 - \mathbf{p}) \hat{g}_-(\bar{\mathbf{k}}_4) \hat{J}_{\bar{\mathbf{k}}_4} \int d\mathbf{k} \frac{C(\mathbf{k}, \mathbf{k} - \mathbf{p})}{D_+(\mathbf{p})} \hat{\psi}_{\mathbf{k},+}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-, \quad (\text{A6.117})$$

where $G_2^{(0)}(\mathbf{k})$ is a suitable function which can be expressed as a sum of graphs with an odd number of propagators, hence it vanishes at $\mathbf{k} = 0$. This implies that $G_2^{(0)}(0) = 0$, so that we can regularize it without introducing any running coupling.

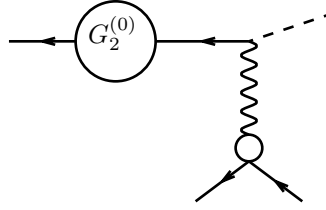


FIG. 8. Graphical representation of (A6.117)

If both $\hat{\psi}_{\mathbf{k},+}^+$ and $\hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-$ in $T_1(\psi)$ are contracted, we get terms of the form

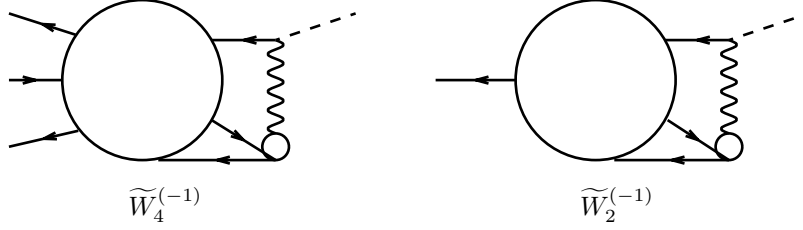
$$\widetilde{W}_{n+1}^{(-1)}(\bar{\mathbf{k}}_4, \mathbf{k}_1, \dots, \mathbf{k}_n) \hat{g}_-(\bar{\mathbf{k}}_4) \hat{J}_{\bar{\mathbf{k}}_4} \prod_{i=1}^n \hat{\psi}_{\mathbf{k}_i}^{\varepsilon_i}, \quad (\text{A6.118})$$

where n is an odd integer. We want to define an \mathcal{R} operation for such terms. There is apparently a problem, as the \mathcal{R} operation involves derivatives and in $\widetilde{W}^{(-1)}$ appears the function $\Delta^{(0,0)}$ of the form (A7.5) and the cutoff function $\tilde{\chi}_M(\mathbf{p})$, with support on momenta of size γ^h . Hence one can worry about the derivatives of the factor $\tilde{\chi}_M(\mathbf{p}) \mathbf{p} D_+(\mathbf{p})^{-1}$. However, as the line of momentum $\bar{\mathbf{k}}_4 - \mathbf{p}$ is necessarily at scale 0 (we are considering terms in which it is contracted), then $|\mathbf{p}| \geq \gamma^{-1} - \gamma^h \geq \gamma^{-1}/2$ (for $|h|$ large enough), so that no bad factors can be produced by the derivatives acting on $\tilde{\chi}_M(\mathbf{p}) \mathbf{p} D_+(\mathbf{p})^{-1}$. We can define the \mathcal{L} operation in the usual way:

$$\begin{aligned} \mathcal{L} \widetilde{W}_4^{(-1)}(\bar{\mathbf{k}}_4, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= \widetilde{W}_4^{(-1)}(\mathbf{0}, \dots, \mathbf{0}), \\ \mathcal{L} \widetilde{W}_2^{(-1)}(\bar{\mathbf{k}}_4) &= \widetilde{W}_2^{(-1)}(\mathbf{0}) + \bar{\mathbf{k}}_4 \partial_{\mathbf{k}} \widetilde{W}_2^{(-1)}(\mathbf{0}). \end{aligned} \quad (\text{A6.119})$$

Note that by parity the first term in the second equation of (A6.119) is vanishing; this means that there are only marginal terms. Note also that the local term proportional to $\hat{J}_{\bar{\mathbf{k}}_4} \hat{\psi}_{\bar{\mathbf{k}}_4,-}^+$ is such that the field $\hat{\psi}_{\bar{\mathbf{k}}_4,-}^+$ can be contracted only at the last scale h ; hence it does not have any influence on the integrations of all the scales $> h$.

If only one among the fields $\hat{\psi}_{\mathbf{k},+}^+$ and $\hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-$ in $T_1(\psi)$ is contracted, we note first that we cannot have terms with two external lines (including $\hat{J}_{\bar{\mathbf{k}}_4}$); in fact in such a case there is an external line with momentum $\bar{\mathbf{k}}_4$ with $\omega = -$ and the other has $\omega = +$; this is however forbidden by global gauge invariance. Moreover, for the same reasons as before, we do not have to worry about the derivatives of the factor $\tilde{\chi}_M(\mathbf{p}) \mathbf{p} D_+(\mathbf{p})^{-1}$,

FIG. 9. Graphical representation of $\widetilde{W}_4^{(-1)}$ and $\widetilde{W}_2^{(-1)}$.

related with the regularization procedure of the terms with four external lines, which have the form

$$\int d\mathbf{k}^+ \hat{\psi}_{\mathbf{k}_1, +}^+ \hat{\psi}_{\mathbf{k}^-, +}^- \hat{\psi}_{\mathbf{k}^- + \bar{\mathbf{k}}_4 - \mathbf{k}_1, -}^+ \hat{g}_-(\bar{\mathbf{k}}_4) \hat{J}_{\bar{\mathbf{k}}_4} \tilde{\chi}_M(\mathbf{k}^+ - \mathbf{k}^-) \hat{g}_-^{(0)}(\bar{\mathbf{k}}_4 - \mathbf{k}^+ + \mathbf{k}^-) \cdot G_4^{(0)}(\mathbf{k}^+, \mathbf{k}_1, \bar{\mathbf{k}}_4 - \mathbf{k}^+ + \mathbf{k}^-) \left\{ \frac{[C_{h,0}(\mathbf{k}^-) - 1] D_+(\mathbf{k}^-) \hat{g}_+^{(0)}(\mathbf{k}^+)}{D_+(\mathbf{k}^+ - \mathbf{k}^-)} - \frac{u_0(\mathbf{k}^+)}{D_+(\mathbf{k}^+ - \mathbf{k}^-)} \right\}, \quad (\text{A6.120})$$

or the similar one with the roles of \mathbf{k}^+ and \mathbf{k}^- exchanged.

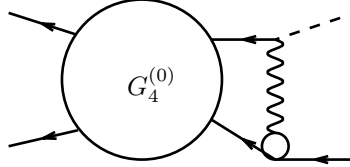


FIG. 10. Graphical representation of a single addend in (A6.120).

The two terms in (A6.120) must be treated differently, as concerns the regularization procedure. The first term is such that one of the external lines is associated with the operator $[C_{h,0}(\mathbf{k}^-) - 1] D_+(\mathbf{k}^-) D_+(\mathbf{p})^{-1}$. We define $\mathcal{R} = 1$ for such terms; in fact, when the $\psi_{\mathbf{k}^-, +}^-$ external line is contracted (and this can happen only at scale h), the factor $D_+(\mathbf{k}^-) D_+(\mathbf{p})^{-1}$ produces an extra factor γ^h in the bound, with respect to the dimensional one. This claim simply follows by the observation that $|D_+(\mathbf{p})| \geq 1 - \gamma^{-1}$ as $\mathbf{p} = \mathbf{k}^+ - \mathbf{k}^-$ and \mathbf{k}^+ is at scale 0, while \mathbf{k}^- , as we said, is at scale h . This factor has the effect that all the marginal terms in the tree path connecting v_0 with the end-point to which is associated the T_1 vertex acquires negative dimension.

The second term in (A6.120) can be regularized as above, by subtracting the value of the kernel computed at zero external momenta, *i.e.* for $\mathbf{k}^- = \bar{\mathbf{k}}_4 = \mathbf{k}_1 = 0$. Note that such local part is given by

$$\int d\mathbf{k}^+ \tilde{\chi}_M(\mathbf{k}^+) \hat{g}_-^{(0)}(\mathbf{k}^+) G_4^{(0)}(\mathbf{k}^+, \mathbf{0}, -\mathbf{k}^+) \frac{u_0(\mathbf{k}^+)}{D_+(\mathbf{k}^+)}, \quad (\text{A6.121})$$

and there is no singularity associated with the factor $D_+(\mathbf{k}^+)^{-1}$, thanks to the support on scale 0 of the propagator $\hat{g}_-^{(0)}(\mathbf{k}^+)$.

A similar (but simpler) analysis holds for the terms contributing to $\bar{\mathcal{V}}_{J,a,2}^{(-1)}$, which contain a vertex of type T_+ or T_- and are of order $\lambda\nu_{\pm}$. Now, the only thing to analyze carefully is the possible singularities associated with the factors $\tilde{\chi}_M(\mathbf{p})$ and $\mathbf{p}D_+(\mathbf{p})^{-1}$. However, since in these terms the field $\hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+$ is contracted, $|\mathbf{p}| \geq \gamma^{-1}/2$, for $|h|$ large enough, a property already used before; hence the regularization procedure can not produce bad dimensional bounds.

We will define \tilde{z}_{-1} and $\tilde{\lambda}_{-1}$, so that

$$\mathcal{L}[\bar{\mathcal{V}}_{J,a,1}^{(-1)} + \bar{\mathcal{V}}_{J,a,2}^{(-1)}](\psi^{[h,-1]}) = \left[\tilde{\lambda}_{-1} Z_{-2}^2 \bar{F}_{\lambda}^{[h,-1]}(\psi^{[h,-1]}) + \tilde{z}_{-1} \hat{\psi}_{\bar{\mathbf{k}}_4,-}^{[h,-1]+} D_-(\bar{\mathbf{k}}_4) \right] \hat{g}_-(\bar{\mathbf{k}}_4) \hat{J}_{\bar{\mathbf{k}}_4}, \quad (\text{A6.122})$$

where we used the definition

$$\bar{F}_{\lambda}^{[h,j]}(\psi^{[h,j]}) = \frac{1}{(M^2)^4} \sum_{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3: C_{h,j}^{-1}(\mathbf{k}_i) > 0} \hat{\psi}_{\mathbf{k}_1,+}^{[h,j]+} \hat{\psi}_{\mathbf{k}_2,+}^{[h,j]-} \hat{\psi}_{\mathbf{k}_3,-}^{[h,j]+} \delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3 - \bar{\mathbf{k}}_4). \quad (\text{A6.123})$$

Note that the fields in the monomial $\bar{F}_{\lambda}^{[h,j]}(\psi^{[h,-1]}) g_-(\bar{\mathbf{k}}_4) \hat{J}_{\bar{\mathbf{k}}_4}$ associated to the coupling $\tilde{\lambda}_{-1}$ have no constraint on the transferred momentum, in particular transferred momenta $|\mathbf{p}| \leq O(\gamma^h)$ are allowed: this deeply distinguish the term associated to $\tilde{\lambda}_{-1}$ from a term like $T_+(\psi)$ in (A6.106): the transferred momentum associated to the fields in $T_+(\psi)$ has instead a lower cutoff $\sim \gamma^h$.

Let us consider now the terms contributing to $\bar{\mathcal{V}}_{J,b,1}^{(-1)}$, that is those where $\hat{\psi}_{\bar{\mathbf{k}}_4-\mathbf{p}}^+$ is not contracted and there is a vertex of type T_1 .

Besides the term of order 0 in λ and ν_{\pm} , equal to $T_1(\psi^{[h,-1]})$, there are the terms containing at least one vertex λ ; among these terms, the only marginal ones (those requiring a regularization) have four external lines (including $\hat{J}_{\bar{\mathbf{k}}_4}$), since the oddness of the propagator does not allow tadpoles. These terms are of the form

$$\sum_{\tilde{\omega}} \int d\mathbf{p} \tilde{\chi}_M(\mathbf{p}) \hat{\psi}_{\mathbf{k}^+, \tilde{\omega}}^+ \int d\mathbf{k}^+ \hat{\psi}_{\mathbf{k}^+ - \mathbf{p}, \tilde{\omega}}^- \hat{\psi}_{\bar{\mathbf{k}}_4 - \mathbf{p}, -}^+ \hat{g}_-(\bar{\mathbf{k}}_4) \hat{J}_{\bar{\mathbf{k}}_4} \left[\hat{F}_{2,+,\tilde{\omega}}^{(-1)}(\mathbf{k}^+, \mathbf{k}^+ - \mathbf{p}) + \hat{F}_{1,+}^{(-1)}(\mathbf{k}^+, \mathbf{k}^+ - \mathbf{p}) \delta_{+,\tilde{\omega}} \right], \quad (\text{A6.124})$$

where $\hat{F}_{2,+,\tilde{\omega}}^{(-1)}$ and $\hat{F}_{1,+}^{(-1)}$ are defined as in (A6.47); they represent the terms in which both or only one of the fields in $\delta\rho_{\mathbf{p},+}$, respectively, are contracted. Both contributions to the r.h.s. of (A6.124) are dimensionally marginal; however, the regularization of $\hat{F}_{1,+}^{(-1)}$ is trivial, as the latter is of the form (A6.55) or the similar one, obtained exchanging \mathbf{k}^+ with \mathbf{k}^- .

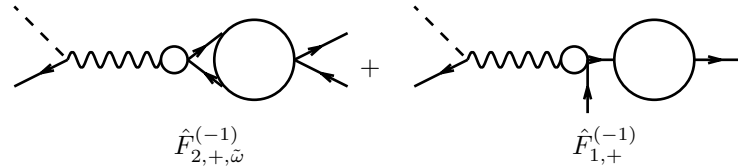


FIG. 11. Graphical representation of (A6.124)

As already discussed above, by the oddness of the propagator in the momentum, $G_+^{(2)}(\mathbf{0}) = 0$, hence we can regularize such term without introducing any local term; the action of \mathcal{R} on it is defined to be the identity.

Moreover, $\hat{F}_{2,+,\tilde{\omega}}^{(-1)}$ satisfies the symmetry properties (A6.51)–(A6.52), so that, defining the action of \mathcal{L} on $\hat{F}_{2,+,\tilde{\omega}}^{(-1)}$ as in (A6.53), we get

$$\mathcal{L}F_{2,+,+}^{-1} = Z_{-1}^{3,+} \quad , \quad \mathcal{L}F_{2,+,-}^{-1} = \frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} Z_{-1}^{3,-} \quad , \quad (\text{A6.125})$$

where $Z_{-1}^{3,+}$ and $Z_{-1}^{3,-}$ are suitable real constants. Hence the local part of the marginal term (A6.124) is, by definition, equal to

$$Z_{-1}^{3,+} T_+(\psi^{[h,-1]}) + Z_{-1}^{3,-} T_-(\psi^{[h,-1]}) \quad . \quad (\text{A6.126})$$

Let us finally consider the terms contributing to $\bar{\mathcal{V}}_{J,b,2}^{(-1)}$, that is those where $\hat{\psi}_{\mathbf{k}_4-\mathbf{p}}^+$ is not contracted and there is a vertex of type T_+ or T_- . If even this vertex is not contracted, we get a contribution similar to (A6.126), with ν_{\pm} in place of $Z_{-1}^{3,\pm}$. Among the terms with at least one vertex λ , there is, as before, no term with two external lines; hence the only marginal terms have four external lines and can be written in the form

$$\int d\mathbf{p} \tilde{\chi}_M(\mathbf{p}) \hat{J}_{\mathbf{k}_4} \hat{g}_-(\mathbf{k}_4) \int d\mathbf{k}^+ \sum_{\tilde{\omega}} \hat{\psi}_{\mathbf{k}^+,\tilde{\omega}}^+ \hat{\psi}_{\mathbf{k}^+-\mathbf{p},\tilde{\omega}}^- \left[\nu_+ G_{+,\tilde{\omega}}^{(0)}(\mathbf{k}^+, \mathbf{k}^+ - \mathbf{p}) + \nu_- \frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} G_{-,\tilde{\omega}}^{(0)}(\mathbf{k}^+, \mathbf{k}^+ - \mathbf{p}) \right] \quad . \quad (\text{A6.127})$$

By using the symmetry property $D_{\omega}(\mathbf{k}) = i\omega D_{\omega}(\mathbf{k}^*)$ discussed in the lines above (A6.50), it is easy to show that $G_{\omega,-\omega}^{(0)}(\mathbf{0}, \mathbf{0}) = 0$. Hence, if we regularize (A6.127) by subtracting $G_{\omega,\tilde{\omega}}^{(0)}(\mathbf{0}, \mathbf{0})$ to $G_{\omega,\tilde{\omega}}^{(0)}(\mathbf{k}^+, \mathbf{k}^+ - \mathbf{p})$, we still get a local term of the form (A6.126).

By collecting all the local term, we can write

$$\mathcal{L}[\bar{\mathcal{V}}_{J,b,1}^{(-1)} + \bar{\mathcal{V}}_{J,b,2}^{(-1)}](\psi^{[h,-1]}) = \nu_{-1,+} T_+(\psi^{[h,-1]}) + \nu_{-1,-} T_-(\psi^{[h,-1]}) \quad , \quad (\text{A6.128})$$

where $\nu_{-1,\omega} = \nu_{\omega} + Z_{-1}^{3,\omega} + G_{\omega,\omega}^{(0)}(\mathbf{0}, \mathbf{0})$. Hence

$$\begin{aligned} \bar{\mathcal{V}}_J^{(-1)}(\psi^{[h,-1]}) &= T_1(\psi^{[h,-1]}) + \nu_{-1,+} T_+(\psi^{[h,-1]}) + \nu_{-1,-} T_-(\psi^{[h,-1]}) + \\ &+ \left[\tilde{\lambda}_{-1} Z_{-2}^2 \bar{F}_{\lambda}^{[h,-1]}(\psi^{[h,-1]}) + \tilde{z}_{-1} \hat{\psi}_{\mathbf{k}_4,-}^{[h,-1]+} D_-(\mathbf{k}_4) \right] \hat{g}_-(\mathbf{k}_4) \hat{J}_{\mathbf{k}_4} + \bar{\mathcal{V}}_{J,R}^{(-1)}(\psi^{[h,-1]}) \quad , \end{aligned} \quad (\text{A6.129})$$

where $\bar{\mathcal{V}}_{J,R}^{(-1)}(\psi^{[h,-1]})$ is the sum of all irrelevant terms linear in the external field J .

Remark. Note that, as already commented after (A6.123), the structure of the field monomials associated to $\tilde{\lambda}_{-1}$ and to $\nu_{-1,+}$ respectively are deeply different, because of the presence of the cutoff function $\tilde{\chi}_M$ in the definition of $T_+(\psi^{[h,-1]})$. This implies that the coupling constant $\tilde{\lambda}_{-1}$ *cannot be included* in the definition of $\nu_{-1,+}$ and is really a different marginal coupling.

A6.8. We now consider the integration of the higher scales. The integration of the field $\psi^{(-1)}$ is done in a similar way; we shall call $\bar{\mathcal{V}}_J^{(-2)}(\psi^{[h,-2]})$ the sum over all terms linear in J . As before, the condition (A6.91) on the external momenta \mathbf{k}_i forbids the presence of vertices of type ϕ , if $h < -1$, as we shall suppose.

The main difference is that there is no contribution obtained by contracting both field variables belonging to $\delta\rho$ in $T_1(\psi)$ at scale -1 , because of (A7.2). It is instead possible to get marginal terms with four external lines (two is impossible), such that one of these fields is contracted at scale -1 . However, in this case, the second field variable will be necessarily contracted at scale h , so that we can put $\mathcal{R} = 1$ for such terms. In fact, after the integration of the last scale field, an extra factor $\gamma^{-(1-h)}$ comes out from a bound similar to that described after (A6.120). Such factor has the effect of automatically regularize these terms, and even the terms containing them as subgraphs.

The terms with a T_1 vertex, such that the field variables belonging to $\delta\rho$ are not contracted, can be treated as in §A6.7, hence do not need a regularization.

It follows that, if the irrelevant part $\bar{\mathcal{V}}_{J,R}^{(-1)}$ were absent in the r.h.s. of (A6.129), then the regularization procedure would not produce any local term proportional to $\bar{F}_\lambda^{[h,-1]}(\psi^{[h,-2]})$, starting from a graph containing a T_1 vertex.

It is easy to see that all other terms containing a vertex of type T_1 or T_\pm can be treated as in §A6.7. Moreover, the support properties of $\hat{g}_-(\bar{\mathbf{k}}_4)$ immediately implies that it is not possible to produce a graph contributing to $\bar{\mathcal{V}}_J^{(-2)}$, containing the \tilde{z}_{-1} vertex. Hence, in order to complete the analysis of $\bar{\mathcal{V}}_J^{(-2)}$, we still have to consider the marginal terms containing the $\tilde{\lambda}_{-1}$ vertex, for which we simply apply the localization procedure defined in (A6.119). We shall define two new constants $\tilde{\lambda}_{-2}$ and \tilde{z}_{-2} , so that $\tilde{\lambda}_{-2}(Z_{-3})^2$ is the coefficient of the local term proportional to $\bar{F}_\lambda^{[h,-1]}(\psi^{[h,-2]})$, while $\tilde{z}_{-2}Z_{-2}\hat{\psi}_{\bar{\mathbf{k}}_4,-}^{[h,-2]+}D_-(\bar{\mathbf{k}}_4)\hat{g}_-(\bar{\mathbf{k}}_4)\hat{J}_{\bar{\mathbf{k}}_4}$ denotes the sum of all local terms with two external lines produced in the second integration step.

The above procedure can be iterated up to scale $h+1$, without any important difference. In particular, for all marginal terms (necessarily with four external lines) such that one of the field variables belonging to $\delta\rho$ in $T_1(\psi)$ is contracted at scale $i \geq j$, we put $\mathcal{R} = 1$. We can do that, because, in this case, the second field variable belonging to $\delta\rho$ has to be contracted at scale h , so that an extra factor $\gamma^{-(i-h)}$ (coming out from a discussion similar to that following (A6.120)) has the effect of automatically regularize their contribution to the tree expansion of $\tilde{G}_+^4(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4)$ (that it similar to that descibed in Chapter 5, with the obvious modifications induced by the presence of new kind of vertices and of a different definition of the \mathcal{L} operator).

Note that, as in the case $j = -1$, there is no problem connected with the presence of the factors $\tilde{\chi}(\mathbf{p})$ and $D_-(\mathbf{p})D_+(\mathbf{p})^{-1}$. In fact, if the field $\hat{\psi}_{\bar{\mathbf{k}}_4-\mathbf{p},-}^+$ appearing in the definition of $T_1(\psi)$ or $T_\pm(\psi)$ is contracted on scale j , each momentum derivative related with the regularization procedure produces the right γ^{-j} dimensional factor, since \mathbf{p} is of order γ^j and the derivatives of $\tilde{\chi}(\mathbf{p})$ are different from 0 only for momenta of order γ^h . If, on the contrary, the field $\hat{\psi}_{\bar{\mathbf{k}}_4-\mathbf{p},-}^+$ is not contracted, then the renormalization procedure is tuned so that $\tilde{\chi}(\mathbf{p})$ and $D_-(\mathbf{p})D_+(\mathbf{p})^{-1}$ are not affected by the regularization procedure.

At step $-j$, we get an expression of the form

$$\begin{aligned} \bar{\mathcal{V}}_J^{(j)}(\psi^{[h,j]}) &= T_1(\psi^{[h,j]}) + \nu_{j,+}T_+(\psi^{[h,j]}) + \nu_{j,-}T_-(\psi^{[h,j]}) + \\ &+ \left[\tilde{\lambda}_j Z_{j-1}^2 \bar{F}_\lambda^{[h,j]}(\psi^{[h,j]}) + \sum_{i=j}^{-1} \tilde{z}_i Z_i \hat{\psi}_{\bar{\mathbf{k}}_4,-}^{[h,j]+} D_-(\bar{\mathbf{k}}_4) \right] \hat{g}_-(\bar{\mathbf{k}}_4) \hat{J}_{\bar{\mathbf{k}}_4} + \bar{\mathcal{V}}_{J,R}^j(\psi^{[h,-1]}), \end{aligned} \quad (\text{A6.130})$$

where $\bar{\mathcal{V}}_{J,R}^j(\psi^{[h,-1]})$ is thought as a convergent tree expansion (under the hypothesis that $\bar{\lambda}_h$ is small enough). Since $Z_{-1} = 1$, this expression is in agreement with (A6.129).

The expansion of $\tilde{G}_+^4(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4)$ is obtained by building all possible graphs with four external lines, which contain one term taken from the expansion of $\bar{\mathcal{V}}_J^{(h)}(\psi^{(h)})$, two terms from $\mathcal{B}_\phi^{(h)}$ and an arbitrary number of terms taken from the effective potential $\mathcal{V}^{(h)}(\psi^{(h)})$. One of the external lines is associated with the free propagator $g_-(\bar{\mathbf{k}}_4)$, the other three are associated with propagators of scale h and momenta $\bar{\mathbf{k}}_i$, $i = 1, 2, 3$.

Remark. With respect to the expansion for G_+^4 , there are three additional quartic running coupling constants, $\nu_{j,+}$, $\nu_{j,-}$ and $\tilde{\lambda}_j$. Note that they are all $O(\lambda)$, despite of the fact that the interaction T_1 has a coupling $O(1)$; this is a crucial property, which follows from the discussion above, implying that either T_1 is contracted at scale 0, or it gives no contribution to the running coupling constants. At a first sight, it seems that now we have a problem more difficult than the initial one; we started from the expansion for G_+^4 , which is convergent if the running coupling λ_j is small, and we have reduced the problem to that of controlling the flow of four running coupling constants, $\nu_{j,+}$, $\nu_{j,-}$, λ_j , $\tilde{\lambda}_j$. However, we will see that, under the hypothesis $\bar{\lambda}_h \leq \varepsilon_1$, also the flow of $\nu_{j,+}$, $\nu_{j,-}$, $\tilde{\lambda}_j$ is bounded. In fact one can use the counterterms ν_+ , ν_- (this is the reason why we introduced them) to impose that $\nu_{j,+}$, $\nu_{j,-}$ are decreasing and vanishing at $j = h$; moreover it

can be verified that the beta functions for $\tilde{\lambda}_j$ and λ_j are identical up to exponentially decaying $O(\gamma^{\tau j})$ terms.

A6.9. Now we will describe the flow of the new effective constants $\nu_{j,\omega}$, $\tilde{\lambda}_j$ and \tilde{z}_j .

First, let us consider $\nu_{j,\pm}$. Note that the definitions of the previous sections imply that there is no contribution to $\nu_{j,\pm}$, coming from trees with a special endpoint of type $\tilde{\lambda}$ or \tilde{z} . Then the contributions to $\nu_{j,\pm}$ either contain a constant $\nu_{k,\pm}$, $k > j$, or an endpoint of type T_1 , that must be on scale 0. Then, by inductively suppose that the size of $\nu_{k,\pm}$, $k > j$ is exponentially small, and using the short memory property, one can show that $\nu_{j,\pm}$ is exponentially small, that is $|\nu_{j,\pm}| \leq c\bar{\lambda}_h \gamma^{\vartheta j}$, for some constants $c, \vartheta > 0$. The formal proof can be done using a fixed point argument, following step by step the analogous analysis used to prove that (A6.70) admits as a solution an exponentially decreasing sequence.

Let us now focus on $\tilde{\lambda}_j$. We start noting that the beta function equation for λ_j can be written as

$$\lambda_{j-1} = \left(\frac{Z_{j-1}}{Z_{j-2}} \right)^2 \lambda_j + \beta_j + \beta_j^{(0)}, \quad (\text{A6.131})$$

where β_j is the sum over the local parts of the trees with at least two endpoints and no endpoint of scale index 0, while $\beta_j^{(0)}$ is the similar sum over the trees with at least one endpoint of scale index 0.

On the other hand we can write

$$\tilde{\lambda}_{j-1} = \left(\frac{Z_{j-1}}{Z_{j-2}} \right)^2 \tilde{\lambda}_j + \tilde{\beta}_j + \tilde{\beta}_j^{(0)} + \tilde{\beta}_j^{(T)} + \tilde{\beta}_j^{(\nu)}, \quad (\text{A6.132})$$

where:

- 1) $\tilde{\beta}_j$ is the sum over the local parts of the trees with at least two endpoints, no endpoint of scale index 0 and one special endpoint of type $\tilde{\lambda}$.
- 2) $\tilde{\beta}_j^{(0)} + \tilde{\beta}_j^{(T)}$ is the sum over the trees with at least one endpoint of scale index 0; $\tilde{\beta}_j^{(0)}$ and $\tilde{\beta}_j^{(T)}$ are, respectively, the sum over the trees with the special endpoint of type $\tilde{\lambda}$ or T_1 .
- 3) $\tilde{\beta}_j^{(\nu)}$ is the sum over the trees with at least two endpoints, whose special endpoint is of type T_{\pm} .

A crucial role in the proof has the following Lemma.

LEMMA A6.1 *Let $\alpha = \tilde{\lambda}_h/\lambda_h$; then if $\bar{\lambda}_h$ is small enough, there exists a constant c , independent of λ , such that $|\alpha| \leq c$ and*

$$|\tilde{\lambda}_j - \alpha \lambda_j| \leq c \bar{\lambda}_h \gamma^{\vartheta j} \quad , \quad h+1 \leq j \leq -1. \quad (\text{A6.133})$$

PROOF - The main point is the remark that there is a one to one correspondence between the trees contributing to β_j and the trees contributing to $\tilde{\beta}_j$. In fact the trees contributing to $\tilde{\beta}_j$ have only endpoints of type λ , besides the special endpoint v^* of type $\tilde{\lambda}$, and the external field with $\omega = -$ and $\sigma = -$ has to belong to P_{v^*} . It follows that we can associate uniquely with any tree contributing to $\tilde{\beta}_j$ a tree contributing to β_j , by simply substituting the special endpoint with a normal endpoint, without changing any label. This correspondence is surjective, since we have imposed the condition that the trees contributing to $\tilde{\beta}_j$ and β_j do not have endpoints of scale index 0. Hence, we can write

$$\left[\left(\frac{Z_{j-1}}{Z_{j-2}} \right)^2 - 1 \right] (\tilde{\lambda}_j - \alpha \lambda_j) + \tilde{\beta}_j - \alpha \beta_j = \sum_{i=j}^{-1} \beta_{j,i} (\tilde{\lambda}_i - \alpha \lambda_i), \quad (\text{A6.134})$$

where, thanks to the “short memory property” and the fact that $Z_{j-1}/Z_{j-2} = 1 + O(\bar{\lambda}_j^2)$, the constants $\beta_{j,i}$ satisfy the bound $|\beta_{j,i}| \leq C \bar{\lambda}_j \gamma^{2\vartheta(j-i)}$, with $\vartheta > 0$.

Among the four last terms in the r.h.s. of (A6.132), the only one depending on the $\tilde{\lambda}_j$ is $\tilde{\beta}_j^{(0)}$, which can be written in the form

$$\tilde{\beta}_j^{(0)} = \sum_{i=j}^{-1} \beta'_{j,i} \tilde{\lambda}_i, \quad (\text{A6.135})$$

the $\beta'_{j,i}$ being constants which satisfy the bound $|\beta'_{j,i}| \leq C \bar{\lambda}_j \gamma^{2\vartheta j}$, since they are related to trees with an endpoint of scale index 0. For the same reasons, we have the bounds $|\tilde{\beta}_j^{(T)}| \leq C \bar{\lambda}_j \gamma^{2\vartheta j}$, $|\beta_j^{(0)}| \leq C \bar{\lambda}_j^2 \gamma^{2\vartheta j}$. Finally, by using the exponential decay of the $\nu_{j,\omega}$, we see that $|\tilde{\beta}_j^{(\nu)}| \leq C \bar{\lambda}_j \bar{\lambda}_h \gamma^{2\vartheta j}$.

We now choose α so that

$$\tilde{\lambda}_h - \alpha \lambda_h = 0, \quad (\text{A6.136})$$

and we put

$$x_j = \tilde{\lambda}_j - \alpha \lambda_j, \quad h+1 \leq j \leq -1. \quad (\text{A6.137})$$

We can write

$$x_{j-1} = x_{-1} + \sum_{j'=j}^{-1} \left[\sum_{i=j'}^{-1} \beta_{j',i} x_i + \sum_{i=j'}^{-1} \beta'_{j',i} (x_i + \alpha \lambda_i) + \tilde{\beta}_{j'}^{(T)} + \tilde{\beta}_{j'}^{(\nu)} - \alpha \beta_j^{(0)} \right]. \quad (\text{A6.138})$$

On the other hand, the condition (A6.136) implies that

$$x_{-1} = - \sum_{j'=h+1}^{-1} \left[\sum_{i=j'}^{-1} \beta_{j',i} x_i + \sum_{i=j'}^{-1} \beta'_{j',i} (x_i + \alpha \lambda_i) + \tilde{\beta}_{j'}^{(T)} + \tilde{\beta}_{j'}^{(\nu)} - \alpha \beta_j^{(0)} \right], \quad (\text{A6.139})$$

so that, if $h+1 \leq j \leq -1$, the x_j satisfy the equation

$$x_j = - \sum_{j'=h+1}^j \left[\sum_{i=j'}^{-1} \beta_{j',i} x_i + \sum_{i=j'}^{-1} \beta'_{j',i} (x_i + \alpha \lambda_i) + \tilde{\beta}_{j'}^{(T)} + \tilde{\beta}_{j'}^{(\nu)} - \alpha \beta_j^{(0)} \right]. \quad (\text{A6.140})$$

We want to show that equation (A6.140) has a unique solution satisfying the bound

$$|x_j| \leq c_0 (1 + |\alpha| \bar{\lambda}_h) \bar{\lambda}_h \gamma^{\vartheta j}, \quad (\text{A6.141})$$

for a suitable constant c_0 , independent of h , if $\bar{\lambda}_h$ is small enough. Hence we introduce the Banach space \mathfrak{M}_ϑ of sequences $\underline{x} = \{x_j, h+1 \leq j \leq -1\}$ with norm $\|\underline{x}\|_\vartheta \stackrel{\text{def}}{=} \sup_j |x_j| \gamma^{-\vartheta j} \bar{\lambda}_h^{-1}$ and look for a fixed point of the operator $\mathbf{T} : \mathfrak{M}_\vartheta \rightarrow \mathfrak{M}_\vartheta$ defined by the r.h.s. of (A6.140). By using the bounds on the various constants appearing in the definition of \mathbf{T} , we can easily prove that there are two constants c_1 and c_2 , such that

$$|(\mathbf{T}\underline{x})_j| \leq c_1 \bar{\lambda}_h (1 + |\alpha| \bar{\lambda}_h) \gamma^{\vartheta j} + c_2 \bar{\lambda}_h \sum_{j'=h+1}^j \sum_{i=j'}^{-1} \gamma^{2\vartheta(j'-i)} |x_i|. \quad (\text{A6.142})$$

Hence, if we take $c_0 = M c_1$, $M \geq 2$, and $\bar{\lambda}_h$ small enough, the ball \mathfrak{B}_M of radius $c_0 (1 + |\alpha| \bar{\lambda}_h)$ in \mathfrak{M}_ϑ is invariant under the action of \mathbf{T} . On the other hand, under the same condition, \mathbf{T} is a contraction in all \mathfrak{M}_ϑ ; in fact, if $\underline{x}, \underline{x}' \in \mathfrak{M}_\vartheta$, then, if $\bar{\lambda}_h$ is small enough,

$$|(\mathbf{T}\underline{x})_j - (\mathbf{T}\underline{x}')_j| \leq c_2 \bar{\lambda}_h^2 \|\underline{x} - \underline{x}'\| \sum_{j'=h+1}^j \sum_{i=j'}^{-1} \gamma^{2\vartheta(j'-i)} \gamma^{\vartheta i} \leq \frac{1}{2} \|\underline{x} - \underline{x}'\| \bar{\lambda}_h \gamma^{\vartheta j}, \quad (\text{A6.143})$$

It follows, by the contraction principle, that there is a unique fixed point in the ball \mathfrak{B}_M , for any $M \geq 2$, hence a unique fixed point in \mathfrak{M}_θ , satisfying the condition (A6.141) with $c_0 = 2c_1$.

To complete the proof, we have to show that α can be bounded uniformly in h . In order to do that, we insert in the l.h.s. of (A6.139) the definition of x_{-1} and we bound the r.h.s. by using (A6.141) and (A6.142); we get

$$|\tilde{\lambda}_{-1} - \alpha\lambda_{-1}| \leq c_3\bar{\lambda}_h + c_4|\alpha|\bar{\lambda}_h^2, \quad (\text{A6.144})$$

for some constants c_3 and c_4 . Since $|\lambda_{-1}| \geq c_5|\lambda|$, $\tilde{\lambda}_{-1} \leq c_6|\lambda|$ and $\bar{\lambda}_h \leq 2|\lambda|$ by the inductive hypothesis, we have

$$|\alpha\lambda_{-1}| \leq |\tilde{\lambda}_{-1}| + c_3\bar{\lambda}_h + c_4|\alpha|\bar{\lambda}_h^2 \Rightarrow |\alpha| \leq (c_6 + 2c_3 + 2c_4|\alpha|\bar{\lambda}_h)/c_5, \quad (\text{A6.145})$$

so that, $|\alpha| \leq 2(c_6 + 2c_3)/c_5$, if $4c_4\bar{\lambda}_h \leq c_5$. ■

We want now to discuss the properties of the constants \tilde{z}_j , $h \leq j \leq -1$, by comparing them with the constants z_j , which are involved in the renormalization of the free measure, see (A6.64). There is a tree expansion for the z_j , which can be written as

$$z_j = \beta_j + \beta_j^{(0)}, \quad (\text{A6.146})$$

where β_j is the sum over the trees without endpoints of scale index 0, while $\beta_j^{(0)}$ is the sum of the others, satisfying the bound $|\beta_j^{(0)}| \leq C\bar{\lambda}_h^2\gamma^{\vartheta j}$. The tree expansion of the \tilde{z}_j can be written as

$$\tilde{z}_j = \tilde{\beta}_j + \tilde{\beta}_j^{(\nu)} + \tilde{\beta}_j^{(0)}, \quad (\text{A6.147})$$

where $\tilde{\beta}_j$ is the sum over the trees without endpoints of scale index +1, such that the special endpoint is of type $\tilde{\lambda}$, $\tilde{\beta}_j^{(\nu)}$ is the sum over the trees whose special endpoint is of type T_\pm , and $\tilde{\beta}_j^{(0)}$ is the sum over the trees with at least an endpoint of scale index 0.

Since there is no tree contributing to $\tilde{\beta}_j^{(0)}$ without at least one λ or $\tilde{\lambda}$ endpoint and since all trees contributing to it satisfy the “short memory property”, by using Lemma A6.1 (which implies that $|\tilde{\lambda}_j| \leq C\bar{\lambda}_h$), we get the bound $|\tilde{\beta}_j^{(0)}| \leq C\bar{\lambda}_h\gamma^{\vartheta j}$. In a similar manner, by using the exponential decay of the constants $\nu_{j,\omega}$, we see that $|\tilde{\beta}_j^{(\nu)}| \leq C\bar{\lambda}_h^2\gamma^{\vartheta j}$.

Let us now consider β_j and $\tilde{\beta}_j$. By an argument similar to that used in the proof of Lemma A6.1, we can write

$$\tilde{\beta}_j - \alpha\beta_j = \sum_{i=j+1}^{-1} \beta_{j,i}(\tilde{\lambda}_i - \alpha\lambda_i), \quad (\text{A6.148})$$

where α is defined as in Lemma A6.1 and $|\beta_{j,i}| \leq C\bar{\lambda}_h\gamma^{2\vartheta j}$. Hence, Lemma A6.1 implies that

$$|\tilde{z}_j - \alpha z_j| \leq C\bar{\lambda}_h\gamma^{\vartheta j}. \quad (\text{A6.149})$$
■

A6.10. In this section we conclude the bound for $\tilde{G}_+^4(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4)$. If we consider the tree expansion for \tilde{G}_+^4 , we realize that there are various classes of trees contributing to it, depending on the type of the special endpoint. Let us consider first the family $\mathcal{T}_{\tilde{\lambda}}$ of the trees with an endpoint of type $\tilde{\lambda}$. These trees have the same structure of those appearing in the expansion of $G_+^4(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4)$, except for the fact that the external (renormalized) propagator of scale h and momentum $\bar{\mathbf{k}}_4$ is substituted with the free propagator $\hat{g}_-(\bar{\mathbf{k}}_4)$. It follows, by using the bound $|\tilde{\lambda}_j| \leq C\bar{\lambda}_h$, that a tree with n endpoint is bounded by $(C\bar{\lambda}_h)^n Z_h^{-1} \gamma^{-4h}$, larger for a factor Z_h with respect to what we need.

Let us now consider the family $\mathcal{T}_{\tilde{z}}$ of the trees with a special endpoint of type \tilde{z} . Given a tree $\tau \in \mathcal{T}_{\tilde{\lambda}}$, we can associate with it the class $\mathcal{T}_{\tilde{z},\tau}$ of all $\tau' \in \mathcal{T}_{\tilde{\lambda}}$, obtained by τ in the following way:

- 1) we substitute the endpoint v^* of type $\tilde{\lambda}$ of τ with an endpoint of type λ ;
- 2) we link the endpoint v^* to an endpoint of type \tilde{z} trough a renormalized propagator of scale h .

Note that $\mathcal{T}_{\tilde{z}} = \cup_{\tau \in \mathcal{T}_{\tilde{\lambda}}} \mathcal{T}_{\tilde{z},\tau}$ and that, if τ has n endpoints, any $\tau' \in \mathcal{T}_{\tilde{z},\tau}$ has $n + 1$ endpoints. Moreover, since the value of $\bar{\mathbf{k}}_4$ has been chosen so that $f_h(\bar{\mathbf{k}}_4) = 1$, $\hat{g}_-^{(h)}(\bar{\mathbf{k}}_4) = Z_{h-1}^{-1} \hat{g}_-(\bar{\mathbf{k}}_4)$; hence it is easy to show that the sum of the values of a tree $\tau \in \mathcal{T}_{\tilde{\lambda}}$, such the special endpoint has scale index $j^* + 1$, and of all $\tau' \in \mathcal{T}_{\tilde{z},\tau}$ is obtained from the value of τ , by substituting $\tilde{\lambda}_{j^*}$ with

$$\Lambda_{j^*} = \tilde{\lambda}_{j^*} - \lambda_{j^*} \frac{\sum_{j=h}^{-1} \tilde{z}_j Z_j}{Z_{h-1}}, \quad (\text{A6.150})$$

see Fig. 12.

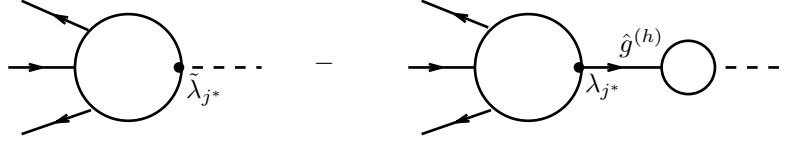


FIG. 12. The resummation of (A6.150).

On the other hand, (A6.149) and the bound $Z_j \leq \gamma^{-C\bar{\lambda}_h^2 j}$, see (5.22), imply that, if $\bar{\lambda}_h$ is small enough

$$\sum_{j=h}^{-1} |\tilde{z}_j Z_j - \alpha z_j Z_j| \leq \sum_{j=h}^{-1} C\bar{\lambda}_h \gamma^{\vartheta j} Z_j \leq C\bar{\lambda}_h. \quad (\text{A6.151})$$

It follows, by using also the bound (A6.133), that

$$\Lambda_{j^*} = \alpha \lambda_{j^*} \left[1 - \frac{\sum_{j=h}^{-1} z_j Z_j}{Z_{h-1}} \right] + \frac{O(\bar{\lambda}_h)}{Z_h}. \quad (\text{A6.152})$$

Moreover, since $Z_{j-1} = Z_j(1 + z_j)$, for $j \in [-1, h]$, and $Z_{-1} = 1$, it is easy to check that

$$Z_{h-1} - \sum_{j=h}^{-1} z_j Z_j = 1. \quad (\text{A6.153})$$

This identity, Lemma A6.1 and (A6.152) imply the bound

$$|\Lambda_{j^*}| \leq C \frac{\bar{\lambda}_h}{Z_h}, \quad (\text{A6.154})$$

which gives us the “missing” Z_h^{-1} factor for the sum over the trees whose special endpoint is of type $\tilde{\lambda}$ or \tilde{z} .

Let us now consider the family \mathcal{T}_{ν} of the trees with a special endpoint of type T_{\pm} . It is easy to see, by using the exponential decay of the $\nu_{j,\omega}$ and the “short memory property”, that the sum over the trees of this class

with $n \geq 0$ normal endpoints is bounded, for $\bar{\lambda}_h$ small enough, by $(C\bar{\lambda}_h)^{n+1} Z_h^{-1} \gamma^{-4h} \sum_{j=h}^{-1} Z_j^{-2} \gamma^{2\vartheta(h-j)} \gamma^{\vartheta j} \leq (C\bar{\lambda}_h)^{n+1} Z_h^{-3} \gamma^{-(4-\vartheta)h}$, which is even better of our needs.

We still have to consider the family \mathcal{T}_1 of the trees with a special endpoint of type T_1 . There is first of all the trivial tree, obtained by contracting all the ψ lines of T_1 on scale h , but its value is 0, because of the support properties of the function $\tilde{\chi}(\mathbf{p})$. Let us now consider a tree $\tau \in \mathcal{T}_1$ with $n \geq 1$ endpoints of type λ . If we call $h_{v_1} = j_1 + 1$ the scale of the vertex T_1 , then the dimensional bound of this tree differs from that of a tree with $n + 1$ normal endpoints contributing to $G_+^4(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4)$ for the following reasons:

- 1) there is a factor Z_h^{-1} missing, because the external (renormalized) propagator of scale h and momentum $\bar{\mathbf{k}}_4$ is substituted with the free propagator $\hat{g}_-(\bar{\mathbf{k}}_4)$;
- 2) there is a factor $|\lambda_{j_1}| Z_{j_1}^2$ missing, because there is no external field renormalization in the $T_1(\psi^{[h,j]})$ contribution to $\tilde{\mathcal{V}}_J^{(j)}(\psi^{[h,j]})$, see (A6.130);
- 3) there is a factor Z_h^{-1} missing, because the factor $\tilde{Z}_{h-1}(\mathbf{k}^-)$ in the r.h.s. of (A7.16) can only be bounded by a constant, because $\tilde{Z}_{h-1}(\mathbf{k}^-)$ is in general different from Z_{h-1} on the support of $f^{(h)}$.

It follows that the sum of the values of all trees $\tau \in \mathcal{T}_1$ with $n \geq 1$ normal endpoints, if $\bar{\lambda}_h$ is small enough, is bounded by $(C\bar{\lambda}_h)^n \gamma^{-4h} \sum_{j_1=h}^0 Z_{j_1}^{-2} \gamma^{2\vartheta(h-j_1)} \leq (C\bar{\lambda}_h)^n \gamma^{-4h} Z_h^{-2}$.

By collecting all the previous bounds, we prove that the bound (A6.95) is satisfied in the case of $H_+^{4,1}$.

Remark. In T_1 and in the Grassmannian monomials multiplying $\nu_{j,+}, \nu_{j,-}$, an external line is always associated to a free propagator $\hat{g}_-(\bar{\mathbf{k}}_4)$; this is due to the fact that, in deriving the Dyson equation, one extracts a free propagator. Then in the bounds there is a Z_h missing (such propagator is not “dressed” in the multiscale integration procedure), and at the end the crucial identity (A6.153) has to be used to “dress” the extracted propagator carrying momentum $\bar{\mathbf{k}}_4$.

A6.11. We finally describe the modifications to the discussion above needed to bound $H_-^{4,1}$.

If we substitute, in the l.h.s. of (A6.103) $H_+^{4,1}$ with $H_-^{4,1}$, we can proceed in a similar way. By using (A6.85), we get

$$\begin{aligned} & \hat{g}_-(\mathbf{k}_4) \frac{1}{L\beta} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) D_+^{-1}(\mathbf{p}) \hat{H}_-^{4,1}(\mathbf{p}; \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4 - \mathbf{p}) = \\ & = \hat{g}_-(\mathbf{k}_4) \frac{1}{L\beta} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{1}{L\beta} \sum_{\mathbf{k}} \frac{C_-(\mathbf{k}, \mathbf{k} - \mathbf{p})}{D_+(\mathbf{p})} < \hat{\psi}_{\mathbf{k},-}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},-}^-; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,-}^-; \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ >^T + \\ & - \nu'_- \hat{g}_-(\mathbf{k}_4) \frac{1}{L\beta} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{1}{L\beta} \sum_{\mathbf{k}} \frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} < \hat{\psi}_{\mathbf{k},-}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},-}^-; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,+}^-; \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ >^T - \\ & - \nu'_+ \hat{g}_-(\mathbf{k}_4) \frac{1}{L\beta} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{1}{L\beta} \sum_{\mathbf{k}} < \hat{\psi}_{\mathbf{k},+}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},+}^-; \hat{\psi}_{\mathbf{k}_1,+}^-; \hat{\psi}_{\mathbf{k}_2,+}^+; \hat{\psi}_{\mathbf{k}_3,-}^-; \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ >^T . \end{aligned} \quad (\text{A6.155})$$

We define $\tilde{G}_-^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ as in (A6.104) with $\widetilde{\mathcal{W}}$ replaced by $\widetilde{\mathcal{W}}_-$ given by

$$\begin{aligned} \widetilde{\mathcal{W}}_- &= \log \int P(d\hat{\psi}) e^{-T_2(\psi) + \nu'_+ T_+(\psi) + \nu'_- T_-(\psi)} e^{-V(\hat{\psi}) + \sum_{\omega} \int d\mathbf{x} [\phi_{\mathbf{x},\omega}^+ \hat{\psi}_{\mathbf{x},\omega}^- + \hat{\psi}_{\mathbf{x},\omega}^+ \phi_{\mathbf{x},\omega}^-]} , \\ T_2(\psi) &= \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{1}{M^2} \sum_{\mathbf{k}} \frac{C_-(\mathbf{k}, \mathbf{k} - \mathbf{p})}{D_+(\mathbf{p})} (\hat{\psi}_{\mathbf{k},-}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},-}^-) \hat{\psi}_{\mathbf{k}_4-\mathbf{p},-}^+ \hat{J}_{\mathbf{k}_4} \hat{g}(\mathbf{k}_4) , \end{aligned} \quad (\text{A6.156})$$

T_+, T_- being defined as in (A6.106). By the analogues of (A6.111), (A6.112) we obtain

$$\tilde{G}_-^4(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4) = \hat{g}_-(\bar{\mathbf{k}}_4) \frac{1}{M^2} \sum_{\mathbf{p}} \tilde{\chi}_M(\mathbf{p}) \frac{\hat{H}_-^{4,1}(\mathbf{p}; \bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4 - \mathbf{p})}{D_+(\mathbf{p})} . \quad (\text{A6.157})$$

The calculation of $\tilde{G}_+^4(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4)$ is done via a multiscale expansion essentially identical to the one of $\tilde{G}_+^4(\bar{\mathbf{k}}_1, \bar{\mathbf{k}}_2, \bar{\mathbf{k}}_3, \bar{\mathbf{k}}_4)$, by taking into account that $\delta\rho_{\mathbf{p},+}$ has to be substituted with

$$\delta\rho_{\mathbf{p},-} = \frac{1}{M^2} \sum_{\mathbf{k}} \frac{C_-(\mathbf{p}, \mathbf{k})}{D_+(\mathbf{p})} (\hat{\psi}_{\mathbf{k},-}^+ \hat{\psi}_{\mathbf{k}-\mathbf{p},-}^-). \quad (\text{A6.158})$$

Let us consider the first step of the iterative integration procedure and let us call again $\bar{\mathcal{V}}_J^{(-1)}(\psi^{[h,-1]})$ the contribution to the effective potential of the terms linear in J . Let us now decompose $\bar{\mathcal{V}}_J^{(-1)}(\psi^{[h,-1]})$ as in (A6.116) and let us consider the terms contributing to $\bar{\mathcal{V}}_{J,a,1}^{(-1)}(\psi^{[h,-1]})$. The analysis goes exactly as before when no one or both the fields $\hat{\psi}_{\mathbf{k},-}^+$ and $\hat{\psi}_{\mathbf{k}-\mathbf{p},-}^-$ of $\delta\rho_{\mathbf{p},-}$ are contracted. This is not true if only one among the fields $\hat{\psi}_{\mathbf{k},-}^+$ and $\hat{\psi}_{\mathbf{k}-\mathbf{p},-}^-$ in $T_2(\psi)$ is contracted, since in this case there are marginal terms with two external lines, which before were absent. The terms with four external lines can be treated as before; one has just to substitute $D_+(\mathbf{k}^-)\hat{g}_+^{(0)}(\mathbf{k}^+)$ with $D_-(\mathbf{k}^-)g_-^{(0)}(\mathbf{k}^+)$ in the r.h.s. of (A6.120), but this has no relevant consequence. The terms with two external lines have the form

$$\int d\mathbf{k}^- \hat{\psi}_{\mathbf{k}_4,-}^+ \hat{g}_-(\bar{\mathbf{k}}_4) \hat{J}_{\bar{\mathbf{k}}_4} \tilde{\chi}_M(\bar{\mathbf{k}}^4 - \mathbf{k}^-) G_1^{(0)}(\mathbf{k}_-) \left\{ \frac{[C_{h,0}^\varepsilon(\bar{\mathbf{k}}_4) - 1] D_-(\bar{\mathbf{k}}_4) \hat{g}_-^{(0)}(\mathbf{k}^-)}{D_+(\bar{\mathbf{k}}^4 - \mathbf{k}^-)} - \frac{u_0(\mathbf{k}^-)}{D_+(\bar{\mathbf{k}}^4 - \mathbf{k}^-)} \right\}, \quad (\text{A6.159})$$

where $G_1^{(0)}(\mathbf{k}_-)$ is a smooth function of order 0 in λ . However, the first term in the braces is equal to 0, since $|\bar{\mathbf{k}}_4| = \gamma^h$ implies that $C_{h,0}^\varepsilon(\bar{\mathbf{k}}_4) - 1 = 0$. Hence the r.h.s. of (A6.159) is indeed of the form

$$\int d\mathbf{k}^- \hat{\psi}_{\mathbf{k}_4,-}^+ \hat{g}_-(\bar{\mathbf{k}}_4) \hat{J}_{\bar{\mathbf{k}}_4} \tilde{\chi}_M(\bar{\mathbf{k}}_4 - \mathbf{k}^-) G_1^{(0)}(\mathbf{k}_-) \frac{u_0(\mathbf{k}^-)}{D_+(\bar{\mathbf{k}}^4 - \mathbf{k}^-)}, \quad (\text{A6.160})$$

so that it can be regularized in the usual way.

The analysis of $\bar{\mathcal{V}}_{J,a,2}^{(-1)}(\psi^{[h,-1]})$ can be done exactly as before. Hence, we can define again $\tilde{\lambda}_{-1}$ and \tilde{z}_{-1} as in (A6.122), with $\tilde{\lambda}_{-1} = O(\lambda)$ and $\tilde{z}_{-1} = O(1)$.

Let us consider now the terms contributing to $\bar{\mathcal{V}}_{J,b,1}^{(-1)}$, that is those where $\hat{\psi}_{\mathbf{k}_4-\mathbf{p}}^+$ is not contracted and there is a vertex of type T_2 . Again the only marginal terms have four external lines and have the form

$$\begin{aligned} & \sum_{\tilde{\omega}} \int d\mathbf{p} \tilde{\chi}_M(\mathbf{p}) \hat{\psi}_{\mathbf{k}^+, \tilde{\omega}}^+ \int d\mathbf{k}^+ \hat{\psi}_{\mathbf{k}^+ - \mathbf{p}, \tilde{\omega}}^+ \hat{\psi}_{\mathbf{k}_4 - \mathbf{p}, -}^+ \hat{g}_-(\bar{\mathbf{k}}_4) \hat{J}_{\bar{\mathbf{k}}_4} \cdot \\ & \frac{D_-(\mathbf{p})}{D_+(\mathbf{p})} \left[F_{2,-,\tilde{\omega}}^{(-1)}(\mathbf{k}^+, \mathbf{k}^+ - \mathbf{p}) + F_{1,-,\tilde{\omega}}^{(-1)}(\mathbf{k}^+, \mathbf{k}^+ - \mathbf{p}) \delta_{-,\tilde{\omega}} \right], \end{aligned} \quad (\text{A6.161})$$

where we are using again a definition analogue to (A6.47). The analysis of the terms $F_{1,-,\tilde{\omega}}^{(-1)}(\mathbf{k}^+, \mathbf{k}^+ - \mathbf{p})$ is identical to the one in §A6.7, while, the symmetry property of the propagator under the replacement $\mathbf{k} \rightarrow \mathbf{k}^*$ implies now that, if we define

$$F_{2,-,\tilde{\omega}}^{-1}(\mathbf{k}^+, \mathbf{k}^-) = \frac{1}{D_-(\mathbf{p})} [p_0 A_{0,-,\tilde{\omega}}(\mathbf{k}^+, \mathbf{k}^-) + p_1 A_{1,-,\tilde{\omega}}(\mathbf{k}^+, \mathbf{k}^-)], \quad (\text{A6.162})$$

and

$$\mathcal{L}F_{2,-,\tilde{\omega}}^{-1} = \frac{1}{D_-(\mathbf{p})} [p_0 A_{0,-,\tilde{\omega}}(0, 0) + p_1 A_{1,-,\tilde{\omega}}(0, 0)], \quad (\text{A6.163})$$

then

$$\mathcal{L}F_{2,-,+}^{-1} = Z_{-1}^{3,-} \frac{D_-(\mathbf{p})}{D_+(\mathbf{p})}, \quad \mathcal{L}F_{2,-,-}^{-1} = Z_{-1}^{3,+}, \quad (\text{A6.164})$$

where $Z_{-1}^{3,+}$ and $Z_{-1}^{3,-}$ are the same real constants appearing in (A6.125). Hence, the local part of the marginal term (A6.161) is, by definition, equal to

$$Z_{-1}^{3,+}T_+(\psi^{[h,-1]}) + Z_{-1}^{3,-}T_-(\psi^{[h,-1]}) . \quad (\text{A6.165})$$

The analysis of $\bar{\mathcal{V}}_{j,b,2}^{(-1)}$ can be done exactly as before, so that we can write for $\bar{\mathcal{V}}_j^{(-1)}$ an expression similar to (A6.129), with $T_2(\psi^{[h,-1]})$ in place of $T_1(\psi^{[h,-1]})$ and $\nu'_{-1,\pm}$ in place of $\nu_{-1,\pm}$.

The integration of higher scales proceed as in §A6.8. In fact, the only real difference we found in the integration of the first scale was in the calculation of the $O(1)$ terms contributing to \tilde{z}_{-1} , but these terms are absent in the case of \tilde{z}_j , $j \leq -2$, because the second term in the expression analogous to (A6.159), obtained by contracting on scale $j < 0$ only one of the fields of $\delta\rho_{\mathbf{p},-}$, is exactly zero. Also in this case, the constants ν'_ω can be chosen again so that the an exponentially decaying bound is satisfied even by the constants $\nu'_{j,\omega}$.

In the analysis of the constants $\tilde{\lambda}_j$ and \tilde{z}_j there is only one difference, concerning the bound (A6.149), which has to be substituted with $\tilde{z}_{-1} - \alpha z_{-1} \leq C$, in the case $j = -1$, but it is easy to see that this has no effect on the bound (A6.154). It follows that the final considerations of §A6.10 stay unchanged and we get for $\tilde{G}_-^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ a bound similar to that proved for $\tilde{G}_+^4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$, so ending the proof of the bound (A6.95).

Appendix A7. The properties of $D_\omega(\mathbf{p})^{-1}C_\omega(\mathbf{k}, \mathbf{k} - \mathbf{p})$.

In this Appendix we describe and collect a number of properties of the operator $D_\omega(\mathbf{p})^{-1}C_\omega(\mathbf{k}, \mathbf{k} - \mathbf{p})$, useful in the analysis of the correction identities. We follow the analogue discussion in section §4.2 of [BM3].

Let us consider the quantity

$$\begin{aligned} \Delta_\omega^{(i,j)}(\mathbf{k}^+, \mathbf{k}^-) &= \frac{C_\omega(\mathbf{k}^+, \mathbf{k}^-)}{D_\omega(\mathbf{p})} \hat{g}_\omega^{(i)}(\mathbf{k}^+) \hat{g}_\omega^{(j)}(\mathbf{k}^-) = \\ &= \frac{1}{Z_{i-1}Z_{j-1}} \frac{1}{D_\omega(\mathbf{p})} \left\{ \frac{\tilde{f}_i(\mathbf{k}^+)}{D_\omega(\mathbf{k}^+)} \left[\frac{\tilde{f}_j(\mathbf{k}^-)}{\chi_{h,0}(\mathbf{k}^-)} - \tilde{f}_j(\mathbf{k}^-) \right] - \frac{\tilde{f}_j(\mathbf{k}^-)}{D_\omega(\mathbf{k}^-)} \left[\frac{\tilde{f}_i(\mathbf{k}^+)}{\chi_{h,0}(\mathbf{k}^+)} - \tilde{f}_i(\mathbf{k}^+) \right] \right\}, \end{aligned} \quad (\text{A7.1})$$

where $\mathbf{p} = \mathbf{k}^+ - \mathbf{k}^-$. The above quantity appears in the expansion for $\hat{H}_{2,1}$ when both the fields of $T_{\mathbf{x},\omega}$ are contracted. Note first that

$$\Delta_\omega^{(i,j)}(\mathbf{k}^+, \mathbf{k}^-) = 0, \quad \text{if } 0 > i, j > h, \quad (\text{A7.2})$$

since $\chi_{h,0}(\mathbf{k}^\pm) = 1$, if $h < i, j < 0$. We will see that this property plays a crucial role; it says that, contrary to what happens for $G^{2,1}$, at least one of the two fermionic lines connected to J must have scale 0 or h .

In the cases in which $\Delta_\omega^{(i,j)}(\mathbf{k}^+, \mathbf{k}^-)$ is not identically equal to 0, since $\Delta_\omega^{(i,j)}(\mathbf{k}^+, \mathbf{k}^-) = \Delta_\omega^{(j,i)}(\mathbf{k}^-, \mathbf{k}^+)$, we can restrict the analysis to the case $i \geq j$.

1) If $i = j = 0$, (A7.1) can be rewritten as

$$\Delta_\omega^{(0,0)}(\mathbf{k}^+, \mathbf{k}^-) = \frac{1}{D_\omega(\mathbf{p})} \left[\frac{f_0(\mathbf{k}^+)}{D_\omega(\mathbf{k}^+)} u_0(\mathbf{k}^-) - \frac{f_0(\mathbf{k}^-)}{D_\omega(\mathbf{k}^-)} u_0(\mathbf{k}^+) \right], \quad (\text{A7.3})$$

where $u_0(\mathbf{k})$ is a C^∞ function such that

$$u_0(\mathbf{k}) = \begin{cases} 0 & \text{if } |\mathbf{k}| \leq 1 \\ 1 - f_0(\mathbf{k}) & \text{if } 1 \leq |\mathbf{k}| \end{cases}. \quad (\text{A7.4})$$

We want to show that

$$\Delta_\omega^{(0,0)}(\mathbf{k}^+, \mathbf{k}^-) = \frac{\mathbf{p}}{D_\omega(\mathbf{p})} \mathbf{S}_\omega^{(0)}(\mathbf{k}^+, \mathbf{k}^-) = \frac{p_0 S_{\omega,0}^{(0)}(\mathbf{k}^+, \mathbf{k}^-) + p S_{\omega,1}^{(0)}(\mathbf{k}^+, \mathbf{k}^-)}{D_\omega(\mathbf{p})}, \quad (\text{A7.5})$$

where $S_{\omega,i}^{(0)}(\mathbf{k}^+, \mathbf{k}^-)$ are smooth functions such that

$$|\partial_{\mathbf{k}^+}^{m_+} \partial_{\mathbf{k}^-}^{m_-} S_{\omega,i}^{(0)}(\mathbf{k}^+, \mathbf{k}^-)| \leq C_{m_+ + m_-}, \quad (\text{A7.6})$$

if $\partial_{\mathbf{k}}^m$ denotes a generic derivative of order m with respect to the variables \mathbf{k} and C_m is a suitable constant, depending on m .

The proof of (A7.5) is trivial if \mathbf{p} is bounded away from 0, for example $|\mathbf{p}| \geq 1/2$. It is sufficient to remark that $\Delta_\omega^{(0,0)}(\mathbf{k}^+, \mathbf{k}^-)$, by the compact support properties of $f_0(\mathbf{k})$, is a smooth function and put $S_{\omega,0}^{(0)} = -i\Delta_\omega^{(0,0)}$, $S_{\omega,1}^{(0)} = \omega\Delta_\omega^{(0,0)}$. If $|\mathbf{p}| \leq 1/2$, we can use the identity

$$\begin{aligned} \Delta_\omega^{(0,0)}(\mathbf{k}^+, \mathbf{k}^-) &= -\frac{f_0(\mathbf{k}^+)u_0(\mathbf{k}^+)}{D_\omega(\mathbf{k}^+)D_\omega(\mathbf{k}^-)} + \\ &+ \frac{\mathbf{p}}{D_\omega(\mathbf{p})} \int_0^1 dt \frac{\mathbf{k}^+ - t\mathbf{p}}{|\mathbf{k}^+ - t\mathbf{p}|} \left[f'_0(\mathbf{k}^+ - t\mathbf{p}) \frac{u_0(\mathbf{k}^+)}{D_\omega(\mathbf{k}^-)} - u'_0(\mathbf{k}^+ - t\mathbf{p}) \frac{f_0(\mathbf{k}^+)}{D_\omega(\mathbf{k}^+)} \right], \end{aligned} \quad (\text{A7.7})$$

from which (A7.6) follows.

2) If $i = 0$ and $h \leq j < 0$, we get

$$\Delta_{\omega}^{(0,j)}(\mathbf{k}^+, \mathbf{k}^-) = -\frac{1}{Z_{j-1}} \frac{\tilde{f}_j(\mathbf{k}^-) u_0(\mathbf{k}^+)}{D_{\omega}(\mathbf{p}) D_{\omega}(\mathbf{k}^-)} + \delta_{j,h} \frac{1}{\tilde{Z}_{h-1}(\mathbf{k}^-)} \frac{f_0(\mathbf{k}^+) u_h(\mathbf{k}^-)}{D_{\omega}(\mathbf{p}) D_{\omega}(\mathbf{k}^+)}, \quad (\text{A7.8})$$

where

$$u_h(\mathbf{k}) = \begin{cases} 0 & \text{if } |\mathbf{k}| \geq \gamma^h \\ 1 - f_h(\mathbf{k}) & \text{if } |\mathbf{k}| \leq \gamma^h \end{cases}. \quad (\text{A7.9})$$

If $j < -1$, the first term in the r.h.s. of (A7.8) vanishes for $|\mathbf{p}| \leq 1 - \gamma^{-1}$, since $u_0(\mathbf{k}^+) \neq 0$ implies that $|\mathbf{k}^+| \geq 1$, so that $|\mathbf{k}^-| = |\mathbf{k}^+ - \mathbf{p}| \geq 1 - (1 - \gamma^{-1}) = \gamma^{-1}$ and, as a consequence, $\tilde{f}_j(\mathbf{k}^-) = 0$. Analogously, the second term in the r.h.s. of (A7.8) vanishes for $|\mathbf{p}| \leq 1 - \gamma^{-1} - \gamma^h$, since $f_0(\mathbf{k}^+) \neq 0$ implies that $|\mathbf{k}^+| \geq 1 - \gamma^{-1}$, so that $|\mathbf{k}^-| \geq \gamma^h$ and, as a consequence, $u_h(\mathbf{k}^-) = 0$. On the other hand, if $j = -1$, because $\tilde{f}_{-1}(\mathbf{k}) u_0(\mathbf{k}) = 0$, we can write

$$u_0(\mathbf{k}^+) \tilde{f}_{-1}(\mathbf{k}^-) = -u_0(\mathbf{k}^+) \mathbf{p} \int_0^1 dt \frac{\mathbf{k}^+ - t\mathbf{p}}{|\mathbf{k}^+ - t\mathbf{p}|} \tilde{f}'_{-1}(\mathbf{k}^+ - t\mathbf{p}). \quad (\text{A7.10})$$

It follows that

$$\Delta_{\omega}^{(0,j)}(\mathbf{k}^+, \mathbf{k}^-) = \frac{\mathbf{p}}{D_{\omega}(\mathbf{p})} \mathbf{S}_{\omega}^{(j)}(\mathbf{k}^+, \mathbf{k}^-), \quad (\text{A7.11})$$

where $S_{\omega,i}^{(j)}(\mathbf{k}^+, \mathbf{k}^-)$ are smooth functions such that

$$|\partial_{\mathbf{k}^+}^{m_0} \partial_{\mathbf{k}^-}^{m_j} S_{\omega,i}^{(j)}(\mathbf{k}^+, \mathbf{k}^-)| \leq C_{m_0+m_j} \frac{\gamma^{-j(1+m_j)}}{\tilde{Z}_{j-1}(\mathbf{k}^-)}, \quad h \leq j < 0. \quad (\text{A7.12})$$

3) If $i = j = h$ we get

$$\begin{aligned} \Delta_{\omega}^{(h,h)}(\mathbf{k}^+, \mathbf{k}^-) &= \frac{1}{D_{\omega}(\mathbf{p})} \frac{1}{\tilde{Z}_{h-1}(\mathbf{k}^+) \tilde{Z}_{h-1}(\mathbf{k}^-)} \\ &\cdot \left[\frac{f_h(\mathbf{k}^+) u_h(\mathbf{k}^-)}{D_{\omega}(\mathbf{k}^+)} - \frac{u_h(\mathbf{k}^+) f_h(\mathbf{k}^-)}{D_{\omega}(\mathbf{k}^-)} \right]. \end{aligned} \quad (\text{A7.13})$$

Since this expression can appear only at the last integration step, it is not involved in any regularization procedure. Hence we only need its size for values of \mathbf{p} of order γ^h or larger. It is easy to see that

$$|\Delta_{\omega}^{(h,h)}(\mathbf{k}^+, \mathbf{k}^-)| \leq \frac{C}{M} \frac{\gamma^{-2h}}{\tilde{Z}_{h-1}(\mathbf{k}^+) \tilde{Z}_{h-1}(\mathbf{k}^-)}, \quad \text{if } |\mathbf{p}| \geq M\gamma^h. \quad (\text{A7.14})$$

4) If $j = h < i < -1$, we get

$$\Delta_{\omega}^{(i,h)}(\mathbf{k}^+, \mathbf{k}^-) = \frac{1}{\tilde{Z}_{h-1}(\mathbf{k}^-) Z_{i-1}} \frac{\tilde{f}_i(\mathbf{k}^+) u_h(\mathbf{k}^-)}{D_{\omega}(\mathbf{p}) D_{\omega}(\mathbf{k}^+)}, \quad (\text{A7.15})$$

which satisfies the bound

$$|\Delta_{\omega}^{(i,h)}(\mathbf{k}^+, \mathbf{k}^-)| \leq \frac{C}{M} \frac{\gamma^{-h-i}}{\tilde{Z}_{h-1}(\mathbf{k}^-) Z_{i-1}}, \quad \text{if } |\mathbf{p}| \geq M\gamma^h. \quad (\text{A7.16})$$

Appendix A8. Proof of Lemma 7.3

Proceeding as in Chapter 6, we first solve the equations for Z_h and $\widehat{m}_h^{(2)}$ parametrically in $\underline{\pi} = \{\pi_h\}_{h \leq h_1^*}$. If $|\pi_h| \leq c|\lambda|\gamma^{(\vartheta/2)(h-h_1^*)}$, the first two assumptions of (7.14) easily follow. Now we will construct a sequence $\underline{\pi}$ such that $|\pi_h| \leq c|\lambda|\gamma^{(\vartheta/2)(h-h_1^*)}$ and satisfying the flow equation $\pi_{h-1} = \gamma^h \pi_h + \beta_\pi^h(\pi_h, \dots, \pi_{h_1^*})$.

A8.1. Tree expansion for β_π^h . β_π^h can be expressed as sum over tree diagrams, similar to those used in §5.5. The main difference is that they have vertices on scales k between h and $+2$. The vertices on scales $h_v \geq h_1^* + 1$ are associated to the truncated expectations (3.30); the vertices on scale $h_v = h_1^*$ are associated to truncated expectations w.r.t. the propagators $g_{\omega_1, \omega_2}^{(1, h_1^*)}$; the vertices on scale $h_v < h_1^*$ are associated to truncated expectations w.r.t. the propagators $g_{\omega_1, \omega_2}^{(2, h_v+1)}$. Moreover the end-points on scale $\geq h_1^* + 1$ are associated to the couplings λ_h or ν_h , as in §5.5; the end-points on scales $h \leq h_1^*$ are necessarily associated to the couplings π_h .

A8.2. Bounds on β_π^h . The non vanishing trees contributing to β_π^h must have at least one vertex on scale $\geq h_1^*$: in fact the diagrams depending only on the vertices of type π are vanishing (they are chains, so they are vanishing, because of the compact support property of the propagator). This means that, by the short memory property: $|\beta_\pi^h| \leq c|\lambda|\gamma^{\vartheta(h-h_1^*)}$.

A8.3. Fixing the counterterm. We now proceed as in Chapter 6 but the analysis here is easier, because no λ end-points can appear and the bound $|\beta_\pi^h| \leq c|\lambda|\gamma^{\vartheta(h-h_1^*)}$ holds. As in Chapter 6, we can formally consider the flow equation up to $h = -\infty$, even if h_2^* is a finite integer. This is because the beta function is independent of $\widehat{m}_k^{(2)}$, $k \leq h_1^*$ and admits bounds uniform in h . If we want to fix the counterterm $\pi_{h_1^*}$ in such a way that $\pi_{-\infty} = 0$, we must have, for any $h \leq h_1^*$:

$$\pi_h = - \sum_{k \leq h} \gamma^{k-h-1} \beta_\pi^k(\pi_k, \dots, \pi_{h_1^*}) . \quad (\text{A8.1})$$

Let $\tilde{\mathfrak{M}}$ be the space of sequences $\underline{\pi} = \{\pi_{-\infty}, \dots, \pi_{h_1^*}\}$ such that $|\pi_h| \leq c|\lambda|\gamma^{-(\vartheta/2)(h-h_1^*)}$. We look for a fixed point of the operator $\tilde{\mathbf{T}} : \tilde{\mathfrak{M}} \rightarrow \tilde{\mathfrak{M}}$ defined as:

$$(\tilde{\mathbf{T}}\underline{\pi})_h = - \sum_{k \leq h} \gamma^{k-h-1} \beta_\pi^k(\pi_k; \dots; \pi_{h_1^*}) . \quad (\text{A8.2})$$

Using that β_π^k is independent from $\widehat{m}_k^{(2)}$ and the bound on the beta function, choosing λ small enough and proceeding as in the proof of Theorem 6.1, we find that $\tilde{\mathbf{T}}$ is a contraction on $\tilde{\mathfrak{M}}$, so that we find a unique fixed point, and the first of (7.16) follows.

A8.4. The flows of Z_h and $\widehat{m}_h^{(2)}$. Once that $\pi_{h_1^*}$ is fixed via the iterative procedure of §A8.3, we can study in more detail the flows of Z_h and $\widehat{m}_h^{(2)}$ given by (7.10). Note that z_h and s_h can be again expressed as a sum over tree diagrams and, as discussed for β_π^h , see §A8.2, any non vanishing diagram must have at least one vertex on scale $\geq h_1^*$. Then, by the short memory property, see §5.11, we have $z_h = O(\lambda^2 \gamma^{\vartheta(h-h_1^*)})$ and $s_h = O(\lambda \widehat{m}_h^{(2)} \gamma^{\vartheta(h-h_1^*)})$ and, repeating the proof of Lemma 6.1, we find the second and third of (7.16).

A8.5. The Lipschitz property (7.17). Clearly, $\pi_{h_1^*}^*(\lambda, \sigma_1, \mu_1) - \pi_{h_1^*}^*(\lambda, \sigma'_1, \mu'_1)$ can be expressed via a tree expansion similar to the one discussed above; in the trees with non vanishing value, there is either a difference

of propagators at scale $h \geq h_1^*$ with couplings σ_h, μ_h and σ'_h, μ'_h , giving in the dimensional bounds an extra factor $O(|\sigma_h - \sigma'_h| \gamma^{-h})$ or $O(|\mu_h - \mu'_h| \gamma^{-h})$; or a difference of propagators at scale $h \leq h_1^*$ (computed by definition at $\hat{m}_h^{(2)} = 0$) with the “corrections” a_h^ω, c_h associated to σ_1, μ_1 or σ'_1, μ'_1 , giving in the dimensional bounds an extra factor $O(|\sigma_1 - \sigma'_1|)$ or $O(|\mu_1 - \mu'_1|)$. Then,

$$\begin{aligned} \left| \pi_{h_1^*}(\lambda, \sigma_1, \mu_1) - \pi_{h_1^*}(\lambda, \sigma'_1, \mu'_1) \right| &\leq c|\lambda| \sum_{k \leq h_1^*} \gamma^{k-h_1^*-1} \\ &\cdot \left[\sum_{h \geq h_1^*} \left(\frac{|\sigma_h - \sigma'_h|}{\gamma^h} + \frac{|\mu_h - \mu'_h|}{\gamma^h} \right) + \sum_{k \leq h \leq h_1^*} (|\sigma_1 - \sigma'_1| + |\mu_1 - \mu'_1|) \right], \end{aligned} \quad (A8.3)$$

from which, using (6.21) and (6.22), we easily get (7.17).

Appendix A9. Independence from boundary conditions.

In this Appendix we prove that the limit $\lim_{M \rightarrow \infty} \frac{1}{M^2} \log \Xi_{AT}^{\gamma_1, \gamma_2}$ considered in §7.5 is independent of the boundary conditions γ_1, γ_2 , in particular we prove that there exist constants $C, c > 0$ such that

$$\left| \log \frac{\Xi_{AT}^{\gamma_1, \gamma_2}}{\Xi_{AT}^-} \right| \leq C e^{-cM\gamma^{h_2^*}}, \quad (\text{A9.1})$$

where we recall that Ξ_{AT}^- is the partition function with antiperiodic boundary conditions in all directions and h_2^* is the scale introduced in §7.4. Note that, if $\gamma^{h_2^*} > 0$, as we are assuming, the propagator $g^{(\leq 1)}$ of the ψ field has a mass $O(\gamma^{h_2^*})$. The analysis of this Appendix is based on the analogue analysis in Appendix G of [M].

By using the construction and the definitions in §3.2–§3.3, we can write

$$\log \Xi_{AT}^{\gamma_1, \gamma_2} = \int P_{\gamma_1}^{(1)}(d\psi^{(1)}, d\chi^{(1)}) P_{\gamma_2}^{(2)}(d\psi^{(2)}, d\chi^{(2)}) e^{\tilde{\lambda}V(\psi, \chi)}, \quad (\text{A9.2})$$

where $P_{\gamma_j}^{(j)}$ are defined as in (4.28) with $P_\sigma(d\psi)$ in the l.h.s. of (4.28) replaced by $P(d\psi)$ and the γ_j -boundary conditions replacing the antiperiodic ones.

Proceeding as in Chapter 4 and 5, we see that $\log \Xi_{AT}^{\gamma_1, \gamma_2}$ can be written as sum of terms of the form $\sum_{\mathbf{x}_1, \dots, \mathbf{x}_n} W_{\gamma_1, \gamma_2}(\mathbf{x}_1, \dots, \mathbf{x}_n)$, with \mathbf{x}_i varying in $[-\frac{M}{2}, \frac{M}{2}] \times [-\frac{M}{2}, \frac{M}{2}]$ and the W are truncated expectations for which a Pfaffian expansion like (4.14) holds. Note that $W(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is periodic with period M in any of its coordinates, for any γ_1, γ_2 ; this follows from the fact that there is an even number of ψ, χ fields associated to any \mathbf{x}_i . Moreover $W(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is translation invariant, so that we can fix one variable to the origin $\mathbf{0}$, for instance \mathbf{x}_1 :

$$\sum_{\mathbf{x}_1, \dots, \mathbf{x}_n} W_{\gamma_1, \gamma_2}(\mathbf{x}_1, \dots, \mathbf{x}_n) = \sum_{\mathbf{x}_1, \dots, \mathbf{x}_n} W_{\gamma_1, \gamma_2}(\mathbf{0}, \mathbf{x}_2, \dots, \mathbf{x}_n). \quad (\text{A9.3})$$

We can write $\sum_{\mathbf{x}_1, \dots, \mathbf{x}_n} W$ as $\sum_{\mathbf{x}_1, \dots, \mathbf{x}_n}^* W + \sum_{\mathbf{x}_1, \dots, \mathbf{x}_n}^{**} W$, where $\sum_{\mathbf{x}_1, \dots, \mathbf{x}_n}^*$ is over the coordinates \mathbf{x}_i varying in $[-\frac{M}{4}, \frac{M}{4}] \times [-\frac{M}{4}, \frac{M}{4}]$ and $\sum_{\mathbf{x}_1, \dots, \mathbf{x}_n}^{**} W$ is the rest. Then $\sum_{\mathbf{x}_1, \dots, \mathbf{x}_n}^{**} W$ is $O(e^{-c\gamma^{h_2^*}M})$, as in W there is surely a chain of propagators exponentially decaying connecting the point $\mathbf{0}$ with a point outside $[-\frac{M}{4}, \frac{M}{4}] \times [-\frac{M}{4}, \frac{M}{4}]$.

On the other hand in $\sum_{\mathbf{x}_1, \dots, \mathbf{x}_n}^* W$ we can use the Poisson summation formula, stating that

$$\frac{1}{M} \sum_{n=0}^{M-1} f\left(\frac{n2\pi}{M} + \frac{\alpha\pi}{M}\right) = \sum_{n \in \mathbb{Z}} \hat{f}(nM) (-1)^{\alpha n}, \quad (\text{A9.4})$$

where f is any smooth 2π -periodic function and $\alpha = 0, 1$. From (A9.4) we find, if $g_{\Lambda_M, \gamma_j}(\mathbf{x})$ is the propagator corresponding to $P_{\gamma_j}(d\psi^{(j)}, d\chi^{(j)})$:

$$\begin{aligned} g_{\Lambda_M, \gamma_j}(\mathbf{x})(\mathbf{x} - \mathbf{y}) &= \sum_{\mathbf{n} \in \mathbb{Z}^2} (\varepsilon'_j)^n (\varepsilon_j)^{n_0} g(\mathbf{x} - \mathbf{y} + \mathbf{n}M) \stackrel{def}{=} \\ &\stackrel{def}{=} g(\mathbf{x} - \mathbf{y}) + \delta g_{\gamma_j}(\mathbf{x} - \mathbf{y}), \end{aligned} \quad (\text{A9.5})$$

where $g(\mathbf{x}) = \lim_{M \rightarrow \infty} g_{\Lambda_M, \gamma_j}(\mathbf{x})$, independent of boundary conditions. Note that the only dependence on boundary conditions in the r.h.s. of (A9.5) is in $\delta g_{\gamma_j}(\mathbf{x} - \mathbf{y})$ and it holds, if $|x - y| \leq \frac{M}{2}$, $|x_0 - y_0| \leq \frac{M}{2}$:

$$|\delta g(\mathbf{x} - \mathbf{y})| \leq C e^{-c_2 \gamma^{h_2^*} M}, \quad (\text{A9.6})$$

with a proper constant c_2 . Hence all the terms in $\sum_{\mathbf{x}_1, \dots, \mathbf{x}_n}^* W$ with at least a $\delta g(\mathbf{x} - \mathbf{y})$ are exponentially bounded, while the part with only $g(\mathbf{x} - \mathbf{y})$ is independent from boundary conditions (and it cancels in the expansion for $\log(\Xi_{AT}^{\gamma_1, \gamma_2} / \Xi_{AT}^-)$). This proves (A9.1).

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